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Research paper

Symmetry properties, conservation laws, reduction and numerical approximations of time-fractional cylindrical-Burgers equation

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ABSTRACT

In this paper the Lie group analysis of the time-fractional cylindrical Burgers equation (time-FCB), which is a fundamental PDE occurring in various areas of applied mathematics, such as fluid mechanics, non-linear acoustics, gas dynamics, traffic flow and etc. is given. For this purpose the Riemann–Liouville derivative is used to implement the Lie algorithm for finding the symmetry operators. A reduced form of the equation is given by using the similarity variables obtained from a symmetry and Erdelyi–Kober operator. In the next step conservation laws are derived via a generalization of Noether's theorem. Finally the Chebyshev wavelets for time-fractional differential equations (FDEs) is applied for solving the considered equation.

1. Introduction

Fractional partial differential equations (FPDEs) are widely used to describe various physical effects and many complex phenomena and the other various field such as: electrochemistry, quantitative biology, engineering, mechanics and etc. Also the use of fractional differentiation for the mathematical modeling of real world has been widespread at the recent years. Some tangible examples are coming in the sequel.

The optical soliton perturbation with fractional temporal evolution is one of the viable means to address a growing problem in telecommunication industry, namely the Internet bottleneck. This problem leads to slow Internet traffic and eventually blockage of the traffic. Several mechanisms have been proposed to address this concern. One of them is to choose time-dependent coefficients of dispersion and non-linearity. But, a better way is to consider fractional temporal evolution. Rezazadeh et. al. have been solved the corresponding fractional equation in [1]. For another example the use of fractional differentiation for the mathematical modeling of real world physical problems such as the earthquake modeling, the traffic flow model with fractional derivatives, measurement of viscoelastic material properties and etc. In this modern era, communication system plays an important role in the world wide society. High frequency communication systems continue to benefit from significant industrial attention, triggered by a host of radio frequency (RF) and microwave (MW) communication systems. These systems use the transmission media for transferring the information carrying signal from one point to another point. In practical application, these equations occur in fractional order, not always in integer order [2].

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There are several analytical methods for solving these kinds of equations. Such as unified method [1], variational approach (a method solving the eigenvalue problem for the higher order differential equations) [3], extended unified method [4] and Lie group method. One of the best method to cognition on these case of equations is Lie group analysis, because one of the important applications of these operators is to find exact solutions from reduced FPDEs [5]. Actually, the symmetry analysis of FDEs and the fractional derivatives are proposed by Gazizov and his collaborators in [6], the other researches have been done valuable studies on FPDEs and gained good results [7–10]. In the most of references, authors worked on only fractional order of the time derivatives, but recently Singla and Gupta have introduced a model for space-time fractional derivative which will be useful for the future works [11].

Riewe [12,13] investigated Lagrangian, Hamiltonian mechanics and formulated a version of the Euler–Lagrange equations. Agrawal continued the study of the fractional Euler–Lagrange equations, [14,15]. Conservation laws and Hamiltonian-type equations for the fractional action principles have also been derived in [16].

The other applications are to achieve conservation laws for FPDEs via Noether's method which is based on the notion of formal Lagrangian. In this study, local conservation laws obtained from non-formal Lagrangian are investigated.

Likewise the case of integer-order PDEs [17], symmetries of FPDEs have been studied extensively. So, the complete algebra of Lie point symmetries are derived for the time-FCB equation

$$D_t^\alpha w + bww_x - a\left(\frac{1}{x}w_x + w_{xx} - \frac{w}{x^2}\right) = 0. \tag{1}$$

Two methods are expressed in order to find conservation laws of Eq. (1) with introducing the Euler–Lagrange operator, Euler–Lagrange equations, formal and non-formal Lagrangian [12,13]. In the first method, components of conservation laws are obtained by using the formal Lagrangian provided by Ibragimov. Density and flux component of conservation laws by using generalization of Noether's theorem and vartional principle could be achieved in the second method.

In the last few years, several numerical methods have been suggested for solving FDEs, [18–20]. Wavelets theory is a relatively new and important theory in the mathematical research. This theory has been applied in a wide range of mathematical and physical science such as optimal control, numerical analysis, signal analysis for waveform representation and segmentations. Based on this theory, an accurate computational method called Chebyshev wavelets with operational matrices of fractional integration is proposed to solve FDEs with initial and boundary conditions. Operational matrices of integration based on proposed method play an important role in the modelling of problems. During the work, a general procedure for forming these matrices is given.

The organization of the article is as follows; Some main results of fractional calculus and definitions are stated in Section 2. In Section 3, the infinitesimal transformations and determining equations of Lie symmetries are given. Section 4 is devoted to the reduction precess of Eq. (1) in terms of Erdelyi–Kober operators [8,21]. In Section 5, Lie point symmetries are applied in order to find fractional component of conservation laws of Eq. (1). Eventually in Section 6, the Chebyshev wavelets computational method for solving the FPDEs for the first time are described. One can see that the applied method gives good approximations for the solutions. Conclusion and results are coming in the sequel.

2. Some main result on fractional calculus

In this section, some important definitions and results of fractional calculus are presented [22,23].

Definition 2.1. The Riemann–Liouville and Caputo fractional partial derivative are defined by,

$$D_t^\alpha u(x,t) = \begin{cases} \dfrac{\partial^n u}{\partial t^n} & \alpha = n \\ \dfrac{1}{\Gamma(n-\alpha)}\dfrac{\partial^n}{\partial t^n}\displaystyle\int_0^t (t-s)^{n-\alpha-1}u(x,s)ds & 0 \le n-1 < \alpha < n, \end{cases} \tag{2}$$

and

$${}^cD_t^\alpha u(x,t) = \begin{cases} \dfrac{\partial^n u}{\partial t^n} & \alpha = n \\ \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^t (t-s)^{n-\alpha-1}\dfrac{\partial^n u(x,s)}{\partial s^n}ds & 0 \le n-1 < \alpha < n. \end{cases} \tag{3}$$

respectively, where ∂_t^n is the usual derivative of order n.

The Laplace transform of Riemann–Liouville and Caputo fractional derivative of order $\alpha > 0$ are

$$L\{D_t^\alpha f(t)\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^k \{D_t^{\alpha-k-1} f(t)\}_{t=0}, \quad n-1 < \alpha < n. \tag{4}$$

$$L\{{}^cD_t^\alpha f(t)\} = s^\alpha F(s) - s^{\alpha-1}F(0), \quad n-1 < \alpha < n. \tag{5}$$

where $L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt$, [23].