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Research paper

Necessary and sufficient conditions of full chaos for expanding Baker-like maps and their use in non-expanding Lorenz maps

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A B S T R A C T

In this work we give necessary and sufficient conditions for a discontinuous *expanding* map f of an interval into itself, made up of N pieces, to be chaotic in the whole interval. For $N = 2$ we consider the class of expanding Lorenz maps, for $N \geq 3$ a class of maps whose internal branches are onto, called Baker-like. We give the necessary and sufficient conditions for a discontinuous expanding map to be chaotic in the whole interval and persistent under parameter perturbations (robust full chaos in short). These classes of maps represent a suitable first return in *non-expanding* Lorenz maps. Thus the obtained conditions can be used to prove robust full chaos in non-expanding Lorenz maps. An example from the engineering application is illustrated.

1. Introduction

We consider a map of an interval I into itself. In the last decades, the condition of persistence of chaos in the whole interval (robust full chaos) has become very important in engineering applications, especially those related to grazing bifurcations [2,3,11,29,36], for security transmissions [24,26,28], as well as in other applied fields, such as physics, economics and social sciences [42,43]. In such applications, the systems are often ultimately described by piecewise smooth maps. In particular, it is known that the three-dimensional ordinary differential equations called Lorenz flows [13] and discontinuity-induced bifurcation [36] can be analyzed by using suitable Poincaré maps, which are often piecewise smooth and discontinuous. An important class of such systems, for which the Poincaré sections are maps with two branches $(N = 2)$, leads to a family of discontinuous maps of an interval, with two increasing branches, called Lorenz maps of Class $\mathcal{A}$ [15,18] and, as we shall recall, already considered by many authors [13,19,22,38,44].

Moreover, particularly important is to investigate the conditions of full chaos, and its robustness, in these kind of maps (i.e. discontinuous and with increasing branches). For the class of expanding Lorenz maps this has been considered in the literature. In fact, a well known sufficient, but not necessary, condition of robust full chaos for an expanding Lorenz map $f(x)$ is $f'(x) > \sqrt{2}$ for any x (seen in [16,17,23,37,40]), while the necessary and sufficient conditions for the case $1 < f'(x) < \sqrt{2}$ can be considered outlined in [15,21].

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Differently, the case of a piecewise smooth map with $N > 2$ branches, has got less attention up to now. Besides the basic results related to the piecewise linear map with constant slope, known as $\beta-$Transformation, considered by many authors (see for example [21]), expanding piecewise monotone maps with $N > 2$ branches have been considered by Li and Yorke in [27], where some relevant properties of the chaotic sets are determined, but not the characterization of full chaos.

Still less attention has been paid to the robustness of the chaotic regime. In many applications it is relevant to get robust chaos, i.e. structurally stable chaos, or persistent under parameter perturbations, following the definition given in [8]. In particular, the occurrence and robustness of full chaos in a Poincaré map which is a *non-expanding Lorenz map*, can be considered as an open problem. Indeed this is a relevant case, also in applications, which motivates the present work. Besides the class of Lorenz maps, we are interested in a particular class of piecewise monotone discontinuous maps with $N > 2$ branches, which is associated with the first return map in Lorenz maps. Their peculiarity is that the internal branches of the first return maps are *onto* the interval, and we call these maps Baker-like. A relevant fact is that even if a Lorenz map is not expanding, its related first return map may be an expanding Lorenz map or Baker-like map, and this allows to get results otherwise difficult to prove.

So, the goal of this work is twofold. A first one is to give the necessary and sufficient conditions for a discontinuous piecewise smooth *expanding* map f of an interval into itself, constituted by N pieces with $N \geq 2$, to be robustly chaotic in the whole interval. As recalled above, for $N = 2$ the map is a Lorenz map of Class $\mathcal{A}$, and this problem has been investigated by other authors, mainly giving sufficient conditions for full chaos. For $N > 2$ we consider a family of expanding Baker-like maps, giving the necessary and sufficient conditions of robust full chaos.

Our second goal is to show how the obtained conditions can be used in the study of a piecewise smooth *non-expanding* Lorenz map of an interval into itself, to prove robust full chaos in non expanding cases.

The paper is organized as follows. In the next section we give definitions and recall some results from the literature. In Section 3 we consider an expanding Lorenz map $f(x)$ in the interval $I = [0, 1]$. The role of the homoclinic bifurcations leading from chaotic intervals to chaos in [0,1] is emphasized, and this leads to the necessary and sufficient condition.

The main result on the first goal is given in Section 4, where we consider an expanding Baker-like map ($N \geq 3$) in [0,1] proving the necessary and sufficient condition for robust full chaos. Moreover, we shall see that a sufficient condition is $f(0) < f(1)$. The results of this section extend to generic expanding maps those in [21] related to the piecewise linear maps called $\beta-$Transformation. The results of Section 4 are used in Section 5 where we consider a family of non-expanding Lorenz maps from the engineering field, showing how to detect conditions of full chaos by use of a suitable first return map, which leads to a Baker-like map. Moreover, the conditions are used to obtain the boundaries of wide regions in the parameter space in which robust full chaos in the non-expanding Lorenz map is persistent. Section 6 concludes.

2. Definitions and preliminary properties

Without loss of generality we consider the unit interval and a one-dimensional discontinuous piecewise $\mathcal{C}^{(1)}$ map $f : [0, 1] \longrightarrow [0, 1]$ with two or more branches. We distinguish between maps with one discontinuity point, i.e. the number of branches is $N = 2$, which lead to the well known class of Lorenz maps, or more branches, $N \geq 3$, and we consider a specific class of maps, called Baker-like (as motivated below).

Definition 1 (Lorenz map)**.** A Lorenz map $x \longmapsto f(x)$ is defined by a function $f : I \longrightarrow I, I = [0, 1]$, with a single discontinuity point $\xi_1 \in (0, 1)$:

$$f(x) = \begin{cases} f_1(x) & \text{if } 0 \leq x < \xi_1 \\ f_2(x) & \text{if } \xi_1 \leq x \leq 1 \end{cases} \tag{1}$$

such that $f_i(x)$ are strictly increasing $\mathcal{C}^{(1)}$ functions in I_i, $i = 1, 2$, with $f_1(\xi_1) = 1$ and $f_2(\xi_1) = 0$, $I_1 = [0, \xi_1]$, $I_2 = [\xi_1, 1]$.

Definition 2 (Baker-like map)**.** A Baker-like map $x \longmapsto f(x)$ is defined by a function $f : I \longrightarrow I$, $I = [0, 1]$ with two or more discontinuity points ξ_i, $0 < \xi_1 \ldots < \xi_{N-1} < 1$, $N \geq 3$:

$$f(x) = \begin{cases} f_1(x) & \text{if } 0 \leq x < \xi_1 \\ f_2(x) & \text{if } \xi_1 \leq x < \xi_2 \\ \vdots & \vdots \\ f_N(x) & \text{if } \xi_{N-1} \leq x \leq 1 \end{cases} \tag{2}$$

where $f_i(x)$ are strictly increasing $\mathcal{C}^{(1)}$ functions defined in $I_i = [\xi_{i-1}, \xi_i]$:

$$f_i : [\xi_{i-1}, \xi_i] \longrightarrow [0, 1], \quad \text{for } 1 \leq i \leq N$$

satisfying $f_i(\xi_i) = 1$ and $f_{i+1}(\xi_i) = 0$ for $1 \leq i \leq N - 1$,

$$\alpha := f_1(0) \in [0, 1) \text{ and } \eta := f_N(1) \in (0, 1]. \tag{3}$$

We call map f in (2) Baker-like because for $f_1(0) = 0$ and $f_N(1) = 1$ it reduces to a Baker map in [0,1] with a finite number N of branches, which is chaotic in the whole interval [0,1] (see [12]).