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The 6th-order weighted ENO schemes for hyperbolic conservation laws

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ABSTRACT

In this work, a 6th-order central-upwind weighted essentially non-oscillatory scheme is devised by additionally introducing a large stencil. The constrained condition of choosing this large stencil is that it should contain upwind information to raise the numerical stability of schemes. The implementation of scheme presented here is as compact as the classical weighted essentially non-oscillatory scheme. Several numerical examples show the robust and high-resolution performances of the scheme which is comparable with the recently developed 6th-order weighted essentially non-oscillatory scheme.

1. Introduction

The distinguishing characteristic of nonlinear hyperbolic conservation laws is that their solutions may contain discontinuities such as shock waves even for smooth initial conditions. Due to this characteristic, the high-order numerical methods constructed by traditional ways could generate spurious oscillations around the discontinuous regions. Thus, to cure these oscillations, some extra techniques are needed to devise the high-order numerical methods. Several high-order numerical methods, e.g. essentially non-oscillatory schemes (ENO) [1,2], weighted ENO schemes (WENO) [3,4], discontinuous Galerkin finite element methods (DG FEM) [5,6] and so on, are capable of producing satisfactory numerical results.

The classical WENO schemes [4], as a kind of the most popular schemes, have attracted a lot of attention due to the feature of easy implementation and high resolution. Several improvements have been developed to decrease the numerical dissipation of the classical WENO schemes. One kind of improvement, e.g. [7,8] and so on, is to modify the weights and distribute a little more weights to the less smooth stencils. Another improvement, e.g. [9–13], is to couple the low-dissipation scheme which is applied on the smooth region and the WENO scheme which is applied on the non-smooth region. Recently, based on the often used 5th-order WENO scheme, several 6th-order central-upwind WENO schemes [14–16] have been developed by adding one more stencil. The idea to devise the 6th-order WENO originates from [14] in which the schemes are designed to maximize order of accuracy and bandwidth, and meanwhile minimize dissipation. By including an additional downwind stencil, the overall central schemes WENO-SYMOO and WENO-SYMBO in [14] are constructed on the basis of optimal order of accuracy and optimal bandwidth efficiency, respectively. However, the order of WENO-SYMBO scheme degenerates even in smooth regions, and the WENO-SYMOO scheme introduces numerical instabilities near the contact surfaces even when only moderate discontinuities are involved. In [15], with the same stencils as [14], the authors define the smoothness indicator of downwind stencil using all six grid values. To achieve the optimal order of accuracy, a reference smoothness indicator τ is devised which is similar to that first presented in [8]. In addition, it is found that this scheme needs extra artificial dissipation to maintain the numerical stability.

In [16], another 6th-order scheme is developed which is analogous with that in [15] but with different smoothness indicator for the newly-introduced downwind stencil and no additional artificial dissipation. We will choose this scheme as comparison since it is recently developed and easily implemented. In numerical simulations, it is found that this scheme cannot tolerate large CFL number. That is, with large CFL number, the scheme generates visible oscillations around discontinuities. Instead of disappearing with increasing cell number, the oscillations show the apparent trend of growth.

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To alleviate (or eliminate) the effect of large CFL number on the stability of scheme, we introduce a different stencil into the classical 5th-order WENO scheme [4]. Since the introduction of downwind stencil may lead to certain numerical instability especially for large CFL number, the stencil introduced should contain upwind information. Using the smoothness indicator consisted of all six grid values to measure the smoothness of the downwind stencil consisted of three grid values, as did in [14–16], is rather abrupt. The smoothness indicator of newly-introduced stencil here is directly measured by the formulation presented by Jiang and Shu in [4]. In general, to make the scheme arrive the 6th-order of accuracy and each stencil contain upwind information, this new stencil needs to be larger than original stencils in theory. The nonlinear weights we get cannot satisfy the sufficient condition of retaining the optimal order, by Taylor analysis, and in fact only the 5th-order of accuracy can be obtained. But, this 5th-order scheme still has the smaller errors and higher resolution when compared with the classical 5th-order WENO scheme [4]. In order to cure the degeneration in accuracy, there are two reliable methods generally: embodying a mapping function [7,17,18] into the construction of weights or devising a reference smoothness indicator [8]. In addition, the implementation of schemes presented here is compact like the schemes in [7] or [8] and no artificial parameter is used to control the resolution of solutions.

The organization of this paper is as follows. In Section 2, the recently developed 6th-order WENO scheme [16] is reviewed and we also investigate the numerical oscillations generated by this scheme under large CFL number. In Section 3, we present the new 6th-order WENO schemes. The mapping function in [18] is used in order to achieve the optimal accuracy. Also, as another way, a reference smoothness indicator is also devised to recover the optimal accuracy. Finally, in Section 4, several common examples are used to demonstrate the numerical performances of the new 6th-order WENO schemes.

2. The 6th-order WENO-CU6 scheme

2.1. Review the construction of the WENO-CU6 scheme

In this section, we briefly review the recently developed 6th-order central-upwind WENO scheme [16] for solving hyperbolic systems of conservation laws

$$u_t + g(u)_x = 0, \quad a \le x \le b, \quad t \ge 0, \tag{1}$$

where $u = (u^1, \cdots, u^m)^T$ and $g(u) = (g^1(u), \cdots g^m(u))^T$. We say the system (1) is hyperbolic if Jacobian matrix $\frac{dg}{du}$ has real and distinct eigenvalues,

$$\lambda^1(u) < \lambda^2(u) < \cdots < \lambda^m(u),$$

for all values of u. First, the computational domain $[a, b]$ is divided into N cells $a = x_{1/2} < x_{3/2} < \cdots < x_{N-1/2} < x_{N+1/2} = b$. Denote the interval $[x_{i-1/2}, x_{i+1/2}]$ by I_i and the length of the cell I_i by Δx_i. For simplicity, we will generally assume a uniform grid, but this is not required.

Solving the conservation laws (1) with a conservative difference approximation to the spatial derivative

$$\frac{du_i(t)}{dt} = -\frac{1}{\Delta x}\left(\hat{g}_{i+1/2} - \hat{g}_{i-1/2}\right), \tag{2}$$

where $u_i(t)$ is the numerical approximation to the column vector $u(x_i, t)$. what to do in the next is how to compute the numerical flux $\hat{g}_{i\pm1/2}$ properly. Following the procedure 2.10 in [19], we use the way of characteristic-wise decomposition to project the relevant vectors u_j and $g(u_j)$ into the local characteristic fields, where j is in a neighborhood of i. Denote by v_j and f_j the obtained vectors in the local characteristic fields. To avoid the presentation of entropy violating solutions, for the each component of characteristic variables, the simple scalar Lax–Friedrichs splitting is used to divide the each component f_j^k of f_j into

$$f_j^k = f_j^{k,+} + f_j^{k,-}, \quad k = 1, \cdots, m \tag{3}$$

where the positive flux $f_j^{k,+} = \frac{1}{2}(f_j^k + \alpha^k v_j^k)$ and negative flux $f_j^{k,-} = \frac{1}{2}(f_j^k - \alpha^k v_j^k)$. The scalar α^k is taken as $\alpha = \max_{1\le i\le N} |\lambda^k(u_i)|$ over the global range of u. The WENO-CU6 reconstruction is used to computed each component $\hat{f}_{i+1/2}^{k,\pm}$ at $x_{i+1/2}$ and then we obtain $\hat{f}_{i+1/2}^{k} = \hat{f}_{i+1/2}^{k,+} + \hat{f}_{i+1/2}^{k,-}$. Finally, transforming the $\hat{f}_{i+1/2}$ in characteristic fields back to the physical space, we obtain the numerical flux $\hat{g}_{i+1/2}$. In general, the difference between the different WENO schemes lies on the reconstruction of $\hat{f}_{i+1/2}^{k,\pm}$. Here we only give the procedure of obtaining the positive numerical flux $\hat{f}_{i+1/2}^{k,+}$, for $\hat{f}_{i+1/2}^{k,-}$ a similar way can be used. For simplicity, we will omit the superscript k in $\hat{f}_{i+1/2}^{k,\pm}$ in the following procedure.

In [16], the positive numerical flux $\hat{f}_{i+1/2}^{+}$ can be achieved by the following three steps:

1. Compute the smoothness indicators of four substencils $S_0 = \{I_{i-2}, I_{i-1}, I_i\}$, $S_1 = \{I_{i-1}, I_i, I_{i+1}\}$, $S_2 = \{I_i, I_{i+1}, I_{i+2}\}$, $S_3 = \{I_{i+1}, I_{i+2}, I_{i+3}\}$ by

$$\beta_r = \sum_{l=1}^{2} \int_{x_{i-1/2}}^{x_{i+1/2}} \Delta x^{2l-1} (p_r^{(l)}(x))^2 dx, \quad r = 0, 1, 2 \tag{4}$$

and

$$\beta_3 = \sum_{l=1}^{5} \int_{x_{i-1/2}}^{x_{i+1/2}} \Delta x^{2l-1} (p_3^{(l)}(x))^2 dx \tag{5}$$

where $p_r(x)$, $r = 0, 1, 2$, are the 2nd-degree interpolation polynomials over the interval $[x_{i+r-5/2}, x_{i+r+1/2}]$ and $p_3(x)$ is the 5th-degree interpolation polynomial over the whole interval $[x_{i-5/2}, x_{i+7/2}]$.

2. Compute the nonlinear weights w_r by the following formulas

$$w_r = \frac{a_r}{a_0 + a_1 + a_2 + a_3}, \quad a_r = d_r\left(C + \frac{\tau}{\beta_r + \varepsilon}\right),$$
$$\tau = \beta_3 - \frac{1}{6}(\beta_0 + 4\beta_1 + \beta_2) \tag{6}$$

and where $d_0 = 1/20$, $d_1 = 9/20$, $d_2 = 9/20$ and $d_3 = 1/20$ are the optimal weights and ε is a parameter used to avoid a division by zero in the denominator and set to be $\varepsilon = 10^{-40}$ in all numerical examples.

3. Compute the convex combination of four substencils,

$$\hat{f}_{i+1/2}^{+} = w_0\hat{f}_{i+1/2}^{+,0} + w_1\hat{f}_{i+1/2}^{+,1} + w_2\hat{f}_{i+1/2}^{+,2} + w_3\hat{f}_{i+1/2}^{+,3} \tag{7}$$

where $\hat{f}_{i+1/2}^{+,r}$ are the 2nd-order numerical fluxes obtained on stencil S_r.

Substituting $\hat{f}_{i+1/2}^{+}$ and $\hat{f}_{i+1/2}^{-}$ obtained by the above procedure into (3), we can get the numerical flux $\hat{f}_{i+1/2}$ at the boundary of cell $x_{i+1/2}$. Similarly, we can get the numerical flux $\hat{f}_{i-1/2}$ at $x_{i-1/2}$ and finally form the scheme (2).

The discretization of the ODEs (2) in time obtained from spatial discretization using the methods of lines approach are solved by the third-order TVD Runge–Kutta method [20,21]

$$\begin{aligned} u^{(1)} &= u^n + \Delta t L(u^n), \\ u^{(2)} &= \frac{3}{4}u^n + \frac{1}{4}u^{(1)} + \frac{1}{4}\Delta t L\left(u^{(1)}\right), \\ u^{n+1} &= \frac{1}{3}u^n + \frac{2}{3}u^{(2)} + \frac{2}{3}\Delta t L\left(u^{(2)}\right), \end{aligned} \tag{8}$$

where L is the spatial discretization operator and u^n is the solution at time t_n.

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