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# A monolithic ALE Newton–Krylov solver with Multigrid-Richardson–Schwarz preconditioning for incompressible Fluid-Structure Interaction

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ABSTRACT

In this paper we study a monolithic Newton–Krylov solver with exact Jacobian for the solution of fully incompressible Fluid-Structure Interaction problems of either steady-state or time-dependent type. Unlike common approaches, the enforcement of the incompressibility conditions both for the fluid and for the solid parts is taken care of by using an inf-sup stable finite element pair, without stabilization terms. The Krylov solver is preconditioned using geometric multigrid with smoothers of Richardson type, in turn preconditioned by additive Schwarz algorithms. The separate solution of fluid or solid operators occurs only at the preconditioning stage of the smoother, thus guaranteeing at each level an accurate interface momentum balance. The definition of the subdomains in the Schwarz smoother is driven by the natural splitting between fluid and solid. For each part and level, the domain is subdivided into a number of minimally overlapping subdomains. Numerical investigations of two and three-dimensional benchmark tests with Newtonian fluids and nonlinear hyperelastic solids are carried out by reporting several performance indices, including condition number estimates. A robust performance of the proposed fully incompressible solver is observed, especially for the more challenging direct-to-steady-state problems.



## 1. Introduction

Fluid-Structure Interaction (FSI) problems are of paramount interest because of a number of ubiquitous applications. To give an idea of such a breadth, we recall examples from aeroelasticity [8,44,58], hydroelasticity [1,43,69], biomechanics [30,39], civil engineering [41,57,64], acoustics [26], and poroelasticity [23]. Several research groups at international level have dedicated their efforts to the study of Fluid-Structure Interactions in universities, research institutes as well as industries. From this interest many conferences and workshops have been organized in the last decades, journals have been established and books published [14,15,30,52]. Furthermore, software projects of both open-source and commercial type have been developed in order to perform numerical simulations of FSI phenomena [7].

Given a certain physical model for the solid and fluid parts, many challenging questions are still nowadays open in the FSI community, ranging from experimental investigations [31,36] to theoretical analysis [17,21,42], from numerical approximation to computational issues. Fluid-Structure Interaction problems are characterized by an intrinsic mathematical challenge, due to the inherent nonlinearity given by a domain that moves as a function of the unknowns. Several choices are possible in terms of the identification of the fluid and solid moving domains (interface tracking or capturing), the definition of the coupling algorithm between fluid and solid (monolithic vs. partitioned, loosely coupled vs. strongly coupled), the discretization procedure (decouple-then-discretize or vice versa), the order between the discretization and the linearization procedures (linearize-then-discretize or vice versa), the linearization scheme (fixed-point, relaxed fixed-point, quasi-Newton, Newton), the choice of the linear solvers and preconditioners.

In this work we focus on the performance of geometric multigrid preconditioners combined with domain decomposition smoothers for monolithic Newton–Krylov solvers of fully incompressible FSI problems of either steady-state or time-dependent type. We intend to highlight the effectiveness of our algorithms in handling two kinds of numerical difficulties concerning FSI

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simulations: the enforcement of incompressibility (both for the fluid and for the solid part), and the computation of direct-to-steady-state solutions. We enforce pure incompressibility conditions both for the fluid and for the solid using inf-sup stable finite element pairs, without introducing slightly-compressible stabilization terms. To the best of our knowledge, this is the first contribution in the literature on the study of this class of preconditioners for the case of full incompressibility both in the fluid and in the solid. Concerning steady-state solutions, our algorithms are able to perform direct-to-steady-state computations. This is an advantage with respect to the more time-consuming practice of using pseudo-time stepping schemes, in which the linear systems to be solved have a better conditioning and a stationary solution is reached as a limit of a time sequence.

Multigrid and domain decomposition ideas draw a lot of attention within the FSI community, as a natural result of the interest emerged both in Computational Fluid Dynamics (CFD) [3,59,62] and Computational Solid Mechanics (CSM) [66]. Multigrid algorithms are taken into account for the solution of large sparse linear systems due to their optimal computational complexity, which can be proven rigorously for model elliptic problems [13]. Typically, multigrid schemes are accelerated by outer Krylov iterations, with the intent to still obtain optimal or nearly optimal complexity schemes, or conversely they are seen as accelerators in the preconditioning stage of existing solvers. On the other hand, domain decomposition methods are very appealing since they allow for the definition of separate subproblems for the solid and fluid parts as well as for an effective parallel implementation. In the realm of monolithic approaches, algorithms for FSI with multigrid and/or domain decomposition techniques are studied in several works, such as [4,6,18,22,32,39,45,49,51,61,67,68]. Either geometric or algebraic multigrid schemes may be considered. Concerning geometric multigrid, the work that is closest to ours and that may be seen as a starting point for our contribution is [51]. Following this work, we also solve the coupled problem in a monolithic manner at each level and we perform partitioning between fluid and solid only at the smoothing level within the multigrid preconditioner. The idea of this approach is to invert smaller matrices with better condition numbers in the smoothing process. However, in [51] the smoothing is partitioned but without using domain decomposition algorithms of Schwarz type within the solid and fluid domains. Also, differently from our work, [51] makes use of pressure stabilization and the Arbitrary Lagrangian Eulerian (ALE) equation is simply taken as a harmonic operator. Ultimately, [51] only deals with time-dependent problems. It is also worth mentioning [39], one of the first works on geometric multigrid for FSI. Here it was shown that a geometric multigrid solver with domain decomposition Vanka-like smoother of Multilevel Pressure Schur Complement (MPSC) type shows better convergence properties with respect to Krylov solvers with ILU preconditioners. In [49,61] this multigrid solver has been placed as preconditioner to Krylov methods. A speedup of geometric multigrid can be achieved with GPU implementations [34]. Although in a partitioned scheme, a geometric multigrid strategy for the fluid part in a finite volume context has been defined in [55,56], with appropriate modifications for moving meshes. On the side of algebraic multigrid algorithms for monolithic FSI, we mention [22,32,45]. In [32] Newton–Krylov solvers with AMG-based preconditioners are defined, where AMG is used either as approximate inverse to the separate field blocks in a block Gauss-Seidel preconditioner, or as a monolithic preconditioner with block Gauss–Seidel smoothing. Ad-hoc strategies are needed for AMG for the elasticity and the Navier–Stokes blocks. In [45] a block-triangular preconditioner for monolithic Newton–Krylov iterations is proposed. Suitable preconditioners are defined for the fluid Jacobian, the solid Jacobian and the pseudo-solid mapping. A spectral analysis is given along with computational tests using AMG for the single blocks. A comparison of either algebraic multigrid or one-level additive Schwarz preconditioners (as studied in [6,67,68]) is carried out in [22].

In order to deal with the several nonlinearities inherent to FSI problems (advection terms, transformations between moving and fixed domains, nonlinear constitutive relations), in this work we use an exact Newton method where we compute the exact Jacobian matrix with automatic differentiation tools provided by the Adept software package [38]. For the purposes of the implementation, automatic differentiation is a very convenient tool that can be exploited with little code modification. Analytic expressions of the exact Jacobian may also be implemented using shape derivative calculus [29,50]. In certain cases it may be more convenient, for simplicity or time performance, to consider the use of approximate Jacobians. In [39] quasi-Newton outer iterations with line search are performed and the Jacobian matrix is computed by a divided difference approach. A quasi-Newton method in which the variation of the fluid domain in the fluid equations is neglected is proposed in [9,10]. In [33] the authors propose a quasi-Newton algorithm based on a reduced model for Fluid-Structure Interaction problems.

For the movement of the solid and fluid domains, we describe the solid motion in a Lagrangian way, while the fluid is observed in Eulerian fashion. We use the ALE approach, which is one of the most popular techniques in the FSI community [24,27,54,63] and it differs from other approaches such as the immersed boundary method [46] or the fully Eulerian approach [25]. A judicious definition of the ALE operator is needed in certain conditions in order to preserve the mesh quality. In this work we follow an approach from [40] to define a convenient linear elastic operator. In our numerical experiments we observed that this approach is very robust at preserving the orientation of the mesh elements, or in other words it avoids mesh entanglement (see also [9]).

The paper is organized as follows. In Section 2 we present the strong and weak formulations of the time-dependent and stationary incompressible FSI problems under investigation. A description of the Jacobian structure, the geometric multigrid preconditioner and the Richardson–Schwarz smoother is given in Section 3. Numerical results of benchmark problems are presented in Section 4. Finally, we draw our conclusions.

## 2. Formulation of the incompressible FSI problem

Here we describe the mathematical formulation of the FSI problem. We first define deformation mappings and displacement fields. Then, we describe the Fluid-Structure Interaction problem in terms of three subproblems with mutual coupling. For more details, we refer the reader to [11,27,29,39].

### 2.1. Deformation mappings and displacement fields

For every time $t \in [0, T]$, let $\Omega_t^f, \Omega_t^s \subset \mathbb{R}^n$ be open sets occupied only by a fluid or by a solid material, with boundaries $\partial\Omega_t^f$ and $\partial\Omega_t^s$ on which outward unit normal fields $\boldsymbol{n}^f$ and $\boldsymbol{n}^s$ are given. In the following, any other symbol endowed with the superscripts $f$ or $s$ will refer either to the fluid or the solid part, respectively. We now define the open set $\Omega_t := \Omega_t^f \cup \Omega_t^s \cup \Gamma_t^i$, which is the current configuration of the overall physical domain, where $\Gamma_t^i$ is the *interface* between fluid and solid. The fluid and solid are immiscible, namely $\Omega_t^f \cap \Omega_t^s = \emptyset$, and they interact through the nonempty interface $\Gamma_t^i = \partial\Omega_t^f \cap \partial\Omega_t^s$. We define the parts of the boundary adjacent only to the fluid or only to the solid as $\Gamma_t^f$ and $\Gamma_t^s$, such that $\partial\Omega_t^f = \Gamma_t^f \cup \Gamma_t^i$ and $\partial\Omega_t^s = \Gamma_t^s \cup \Gamma_t^i$ (Fig. 1).

For every domain $D_t \subset \mathbb{R}^n$ (which may change in time), we also define the cylinder $Q_D = \{(\boldsymbol{x}, t) \text{ s.t. } \boldsymbol{x} \in D_t, t \in [0, T]\}$.