Probabilistic Engineering Mechanics 53 (2018) 143–153

Contents lists available at ScienceDirect

Probabilistic Engineering Mechanics

journal homepage: www.elsevier.com/locate/probengmech

Probabilistic assessment of performance under uncertain information using a generalized maximum entropy principle

Alice Cicirello [a,*], Robin S. Langley [b]

[a] Department of Engineering Science, University of Oxford, Parks Road, Oxford OX1 3PJ, United Kingdom
[b] Department of Engineering, University of Cambridge, Trumpington Street, Cambridge CB2 1PZ, United Kingdom

ARTICLE INFO

Keywords:
Maximum entropy
Uncertain probability density function
Inequality constraints on statistical moments
Bounds on failure probability
Bounds on performance metric

ABSTRACT

When information about a distribution consists of statistical moments only, a self-consistent approach to deriving a subjective probability density function (pdf) is Maximum Entropy. Nonetheless, the available information may have uncertainty, and statistical moments maybe known only to lie in a certain domain. If Maximum Entropy is used to find the distribution with the largest entropy whose statistical moments lie within the domain, the information at only a single point in the domain would be used and other information would be discarded. In this paper, the bounded information on statistical moments is used to construct a family of Maximum Entropy distributions, leading to an uncertain probability function. This uncertainty description enables the investigation of how the uncertainty in the probabilistic assignment affects the predicted performance of an engineering system with respect to safety, quality and design constraints. It is shown that the pdf which maximizes (or equivalently minimizes) an engineering metric is potentially different from the pdf which maximizes the entropy. The feasibility of the proposed uncertainty model is shown through its application to: (i) fatigue failure analysis of a structural joint; (ii) evaluation of the probability that a response variable of an engineering system exceeds a critical level, and (iii) random vibration.

1. Introduction

In engineering, several parameters of a computational model (such as geometry, material properties, loadings, boundary conditions, and structural joints) used to investigate the behaviour of a system may not be known precisely, and yet an engineering assessment of a design must nonetheless be performed. These uncertainties can be modelled in a parametric [1,2] or non-parametric way [2–5], or a combination of both [6–9]. While parametric approaches consider specific physical properties of the system to be uncertain, the non-parametric approaches account for the uncertainty effects at a higher level. Parametric uncertainty models can be probabilistic [1] or non-probabilistic (such as intervals [10], convex [11], and fuzzy [12]) and they are used in conjunction with a computational model to compute the effects of the uncertainties on the response.

The most widely used parametric uncertainty description is the probabilistic one based on a specified probability density function (pdf). This description requires a large amount of data if the pdf is constructed using a frequentist view, or it may be interpreted as a statement of belief based on expert opinion, as in the subjective approach to probability theory. The more common frequentist approach is concerned with the outcome of experiments performed (hypothetically or in reality) on large ensembles of systems; these ensembles may either be real (for example cars from a production line), or virtual but realizable in principle (such as an ensemble of manufactured satellites, when only one satellite may actually be built). In contrast, with the subjective approach, no ensemble is necessarily involved. The pdf is interpreted as a statement of belief, rather than a frequentist statement, meaning that the analyst can specify a pdf in the absence of large quantities of data. The frequentist and subjective views can be roughly aligned with the notion of aleatory and epistemic uncertainty: aleatory uncertainty is an irreducible uncertainty due to an inherent variability of the system parameters, while epistemic uncertainty is reducible, being associated to a lack of knowledge of the actual values of the parameters which are fixed.

In the frequentist case there is often insufficient data to empirically determine the pdf, due to cost or time constraints, and it may not be possible to take measurements if the structure does not yet exist. Similarly, in the subjective case, the analyst may have uncertainties in belief,

* Corresponding author.
E-mail addresses: alice.cicirello@eng.ox.ac.uk (A. Cicirello), rsl21@eng.cam.ac.uk (R.S. Langley).

http://dx.doi.org/10.1016/j.probengmech.2017.07.006
Received 23 March 2017; Received in revised form 8 June 2017; Accepted 25 July 2017
Available online 5 August 2017
0266-8920/

meaning that the specified pdf is itself subject to doubt. Alternative uncertainty models have been developed by introducing uncertainty in the assignment of the parameters of a probability density function (pdf), and/or the pdf itself. These models are broadly referred to as "imprecise probability approaches". The idea of specifying upper and lower bounds on an imprecisely known probability of an event was introduced about 100 years ago by Boole [13] and Keynes [14]. Later, Walley [15] and Weichselberger [16] developed generalized probability theories. The imprecise probability approaches which are most widely used can be broadly classified into five groups: (i) Probability boxes [17,18]; (ii) Possibility theory [19,20], (iii) Evidence theory [21–25], (iv) Imprecise pdf descriptions based on the specification of interval constraints on the expectation of functions of the uncertain variable, such as the approach proposed by Utkin and co-workers [26,27]; (v) Probability and cumulative density functions (pdf and cdf) with non-probabilistic parameters (i.e. interval, convex, fuzzy descriptions), such as: (a) Parameterized P-Box [28]; (b) Fuzzy probability theory (also known as Fuzzy randomness) [29–31]; (c) Fuzzy random variable [32,33]; (d) First Order Reliability Method (FORM) [1] approaches which employ pdfs with one [34] or two [35] bounded parameters (mean, variance or another distribution parameter); (e) Reliability models based on imprecise Bayesian inference models [36]; (f) Interval importance sampling methods combined with specified pdf with bounded parameters [37].

Another way of dealing with limited information about a distribution is Maximum Entropy [38]. Maximum Entropy is a well-established approach to deriving a subjective probability density function (pdf) using statistical moments information only. However, there might be little confidence on statistical moments estimated from a small data set. Moreover, if no data is available and the pdf is interpreted as a statement of belief, the analyst may have uncertainties in belief and might prefer not to specify exact statistical moments. Therefore, statistical moments might be known only to lie in a certain domain, rather than being precisely known. In this case Maximum Entropy would select a unique distribution which maximizes the entropy and whose statistical properties are within the statistical moment domain [38,39]. However, in this process some of the initial information is lost, since only one point of the statistical moment domain would be considered and other points, to which correspond different pdfs, are discarded. However, the pdf which maximizes (or equivalently minimizes) an engineering metric is potentially different from the pdf which maximizes the entropy.

A new approach to uncertainty modelling which is based on a generalization of maximum entropy theory is presented in this paper, and applied to a number of engineering examples. With this approach, when the statistical moments are known only to lie in a certain domain, instead of selecting the pdf which maximize the entropy, a family of Maximum Entropy distributions is constructed. This is achieved by representing the pdf of the vaguely known variable as the exponential of a linear combination of functions of the uncertain variable and bounded parameters. This form is equivalent to the Maximum entropy distribution, where the Lagrange multipliers (which are constant values) are substituted by bounded parameters, leading to a set of pdfs. These bounded parameters are referred to as basic variables, defined as having any form of distribution lying between certain bounds, encompassing at the extreme a delta function at any point between the bounds. A mapping procedure is devised to convert bounded information on the statistical moments into bounds on the basic variables. With this approach a bounded response description is then obtained by maximizing (minimizing) a response metric over the set of pdfs to: (i) establish the effects of the imprecisely known pdf on the response; and (ii) identify of the worst case scenario (e.g. the highest failure probability expected).

The present approach has similarities to that proposed by Utkin [26,27], in that the information considered is a set of constraints on the statistical moments. However, in [26,27] the set of moments yield to many possible distributions with no consideration of Entropy, and a set of complex optimization problems is performed to yield bounds on the expectation of the response or on the reliability of the system, which restrict its application to simple problems. Instead, within the proposed approach, the specified set of constraints are used to: (i) identify the form of the Maximum Entropy distribution, and to (ii) yield bounds on the basic variables of the pdf to construct a set of maximum entropy pdfs.

As described above, the main aim of the present paper is to present a generalization of the Maximum Entropy principle given uncertain statistical information, leading to a family of probability distributions that can be used to make engineering judgements. The theory behind this approach is presented in Section 2, with a numerical example of the treatment of bounded information in Section 2.2.4. Attention is then turned to three engineering applications, namely the fatigue failure of a structural joint (Section 3.1), the overstress failure of a structural joint (Section 3.2), and the random vibration of an oscillator (Section 3.3). Aspects of theory relating specifically to the example applications are contained in the relevant subsections, to emphasize the fact that the theory presented in Section 2 is general, and not directed at any specific example.

2. Generalized Maximum Entropy distribution under uncertain statistical information

In Section 2.1 the procedure for deriving a probability density function with the Maximum Entropy principle is first reviewed. The approach is then generalized to account for uncertainty in the statistical moments in Section 2.2.

2.1. Review of Maximum Entropy

The principle of Maximum Entropy [38] allows the construction of a subjective pdf $p(x)$ of an uncertain variable x [38] which incorporates the current state of knowledge by maximizing the relative entropy subject to constraints representing the available information.

The relative entropy, that is the amount of uncertainty in the probability distribution $p(x)$, is given by [38]:

$$H = -\int_{-\infty}^{+\infty} p(x)\log\left(\frac{p(x)}{t(x)}\right)\mathrm{d}x \tag{1}$$

where $t(x)$ is a reference pdf (which is also known as the prior distribution) introduced to allow the entropy to be frame invariant [38,40].

The available information regarding the statistics of the variable x is expressed in terms of n equality constraints on the statistical expectations in the form:

$$\mathrm{E}\left[f_j(x)\right] = \int_{-\infty}^{+\infty} f_j(x)\,p(x)\,\mathrm{d}x \quad j = 2,3,\ldots,n \tag{2}$$

where $f_j(x)$ are specified functions of x , and $\mathrm{E}\left[f_j(x)\right]$ is the statistical expectation of $f_j(x)$. If $f_j(x) = x$ then the constraints are specified on the mean value, alternatively if $f_j(x) = x^2$ they are specified on the second moment. The function $f_j(x)$ can be also defined as an interval of possible values that the uncertain variable may take, i.e. $f_j(x) = [b, c]$; in this case the constraints corresponds to the probability of finding x within those bounds.

This constrained maximization problem (maximizing Eq. (1) subject to Eq. (2)) can be solved by using the method of Lagrange multipliers [38], which is based on transforming the original constrained optimization problem into an unconstrained dual optimization problem in the form:

$$-\int_{-\infty}^{+\infty} p(x)\log\left(\frac{p(x)}{t(x)}\right)\mathrm{d}x + \sum_{j=1}^{n}\lambda_j\left\{\int_{-\infty}^{+\infty} f(x)\,p(x)\,\mathrm{d}x - \mathrm{E}\left[f_j(x)\right]\right\} \tag{3}$$

where λ_j are the n Lagrange multipliers. The maximization of the functional in Eq. (3) is then obtained using the calculus of variations,