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Global well-posedness of the two-dimensional Benjamin equation in the energy space

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ABSTRACT

In this paper, the global well-posedness of the Cauchy problem for the two-dimensional Benjamin equation

$$\begin{cases}\partial_t u + \partial_x^3 u - \epsilon \mathcal{H}\partial_x^2 u - \partial_x^{-1}\partial_y^2 u + \partial_x(u^2/2) = 0,\\ u(x,y,0) = \phi(x,y)\end{cases}$$

in the energy space $E^1 = \{u : \|u\|_{L^2} + \|\partial_x u\|_{L^2} + \|\partial_x^{-1}\partial_y u\|_{L^2} < \infty\}$ is obtained.

1. Introduction

This paper is mainly concerned with the global well-posedness of the Cauchy problem for the two dimensional Benjamin equation (2D Benjamin)

$$\begin{cases}\partial_t u + \partial_x^3 u - \epsilon \mathcal{H}\partial_x^2 u - \partial_x^{-1}\partial_y^2 u + \partial_x(u^2/2) = 0,\\ u(x,\, y,\, 0) = \phi(x,\, y),\end{cases} \tag{1.1}$$

where $u = u(x,y,t), (x,y) \in \mathbb{R}^2$, $t \in \mathbb{R}$, $\epsilon = \pm 1$, $\mathcal{H}$ is the Hilbert transform in the x axis defined by $\widehat{\mathcal{H}u}(\xi) = -i\mathrm{sgn}(\xi)\hat{u}(\xi)$. The inverse derivative operator ∂_x^{-1} is defined by its Fourier transform $\widehat{\partial_x^{-1}}(\xi) = \frac{1}{i\xi}$. The 2D Benjamin arises as a high dimensional extension of the Benjamin equation. The latter models the dispersive wave motion of weakly nonlinear long waves in a two fluid system where the interface is subject to capillarity and the lower fluid is very deep (see [1,2]). The 2D Benjamin equation allows for weak spatial variations transverse to the propagation direction, and can be formally derived by a standard weakly nonlinear long wave expansion (see [3]).

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The Benjamin equation has been extensively studied (see [4–8]). It can be written as

$$\begin{cases}\partial_t u-\gamma\partial_x u+\alpha\mathcal{H}\partial_x^2 u+\beta\partial_x^3 u+\partial_x(u^2)=0, & (x,t)\in\mathbb{R}\times\mathbb{R},\\ u(x,0)=u_0(x), & x\in\mathbb{R}.\end{cases}\tag{1.2}$$

The Benjamin equation is viewed as a combination of the Benjamin–Ono (BO) equation with the Korteweg–de Vries (KdV) equation. The former requires $\alpha\neq 0$ and $\beta=0$ in (1.2) and the latter asks for $\alpha=0$ and $\beta\neq 0$. This slight variation brings tremendous difference. The dispersive term in the BO equation is too weak to obtain the well-posedness by Picard iteration, so the wellposed results in low regularity spaces are not as good as those of KdV problem (see [9–11]). The main obstruction to simply using bilinear estimates in some $X^{\sigma,b}$ space (in a way similar to the case of the KdV equation in [12] or nonlinear wave equations in [13]) is the lack of control of the interaction between very high and very low frequencies of solutions. T. Tao proved that the BO equation initial value problem is globally well-posed in $H^1(\mathbb{R})$ [14]. A. Ionescu and C. Kenig obtained its global well-posedness in H^σ, $\sigma\geq 0$ [15]. The KdV equation, without the competition between the Benjamin–Ono term and the third order derivative term, has wellposedness in lower regularity spaces. C. Kenig, G. Ponce and L. Vega obtained the local well-posedness in classical Sobolev spaces of negative indices $H^\sigma(\mathbb{R})$, $\sigma>-3/4$ [16] and L. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao extended it to a global result (see [17]). For the endpoint $\sigma=-3/4$, M. Christ, J. Colliander and T. Tao obtained the local well-posedness (see [18]). Z. Guo [19] and N. Kishimoto [20] extended it to the global well-posedness independently. For $\sigma<-3/4$, C. Kenig, G. Ponce and L. Vega [21], M. Christ, J. Colliander and T. Tao [18] got ill-posedness. For the Benjamin equation (1.2) with both the Benjamin–Ono term and the third order derivative term shares the same results with KdV in well-posedness (for more details see [22–25]). 2D Benjamin is viewed as a combination of the BO and Kadomtsev–Petviashvili (KP) equation. This motivates us to consider 2D Benjamin and ask whether it shares the wellposedness results with the KP equation.

The KP equation is written as

$$\partial_t u+\partial_x^3 u-\partial_x^{-1}\partial_y^2 u+\partial_x(u^2/2)=0.\tag{1.3}$$

There are actually two types of KP equations, KP-I and KP-II. Here we only mention the first type which has direct relation with the 2D Benjamin equation (1.1). As the BO equation, KP-I has been shown in [26] and [27] that it is badly behaved with respect to Picard iterative methods in standard Sobolev spaces, since the flow map fails to be real-analytic at the origin in these spaces. If we lose the real-analytic flow restriction, there still are some wonderful results. We would like to mention the excellent work of A. Ionescu, C. Kenig and D. Tataru [28]. They set up the global well-posedness of KP-I in the energy space. Here the energy space can be defined by

$$E^1=\Big\{u\in\mathcal{S}'(\mathbb{R}^2):\|u\|_{E^1}=\big|\big|(1+|\xi|+|\xi|^{-1}|\eta|)\hat{u}(\xi,\eta)\big|\big|_{L^2}<\infty\Big\}.\tag{1.4}$$

This space comes from the conservation of the momentum and energy of KP-I. In their paper, they extensively developed the modified energy space in the so called “short-time” Bourgain spaces which became a powerful method to set up the well-posedness in many nonlinear PDEs.

The 2D Benjamin equation possesses the following two conservation laws:

$$\widetilde{E}^0(u(t))=\int_{\mathbb{R}^2}u^2(x,y,t)dxdy=\int_{\mathbb{R}^2}u^2(x,y,0)dxdy=\widetilde{E}^0(u(0)),$$

and

$$\begin{aligned}\widetilde{E}^1(u(t))&=\frac{1}{2}\int_{\mathbb{R}^2}\left[(\partial_x u)^2\pm\big|\,|D_x|^{1/2}u\big|^2+(\partial_x^{-1}\partial_y u)^2-\frac{u^3}{3}\right](x,y,t)dxdy\\ &=\frac{1}{2}\int_{\mathbb{R}^2}\left[(\partial_x u)^2\pm\big|\,|D_x|^{1/2}u\big|^2+(\partial_x^{-1}\partial_y u)^2-\frac{u^3}{3}\right](x,y,0)dxdy=\widetilde{E}^1(u(0)).\end{aligned}$$