Nonlinear Analysis: Real World Applications 45 (2019) 866–876

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

Continuity properties of the data-to-solution map for the two-component higher order Camassa–Holm system

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ARTICLE INFO

Article history:
Received 26 March 2018
Received in revised form 19 June 2018
Accepted 16 August 2018
Available online 31 August 2018

Keywords:
Two-component higher order Camassa–Holm system
Cauchy problem
Well-posedness
Hölder continuity

ABSTRACT

This work studies the Cauchy problem of a two-component higher order Camassa–Holm system, which is well-posed in Sobolev spaces $H^s(\mathbb{R}) \times H^{s-2}(\mathbb{R})$, $s > \frac{7}{2}$ and its solution map is continuous. We show that the solution map is Hölder continuous in $H^s(\mathbb{R}) \times H^{s-2}(\mathbb{R})$ equipped with the $H^r(\mathbb{R}) \times H^{r-2}(\mathbb{R})$-topology for $1 \leq r < s$, and the Hölder exponent is expressed in terms of s and r.

1. Introduction

In this paper, we consider the Cauchy problem of the following two-component higher order Camassa–Holm system

$$\begin{cases} m_t = \alpha u_x - b u_x m - u m_x - \kappa \rho \rho_x, \quad m = Au, \\ \rho_t = -u\rho_x - (b-1)u_x \rho, \quad b \in \mathbb{R} \setminus \{1\}, \\ \alpha_t = 0, \end{cases} \tag{1.1}$$

where $Au = (1-\partial_x^2)^\sigma u$ with $\sigma > 1$, and b, κ are real parameters. Eq. (1.1) was proposed by Escher and Lyons [1], in which they showed that the system corresponds to a metric induced geodesic flow on the infinite dimensional Lie group $\mathrm{Diff}^\infty(\mathbb{S}^1) \circledS C^\infty(\mathbb{S}^1) \times \mathbb{R}$ and admits a global solution in $C^\infty([0,\infty); C^\infty(\mathbb{S}^1) \oplus C^\infty(\mathbb{S}^1))$ with smooth initial data in $C^\infty(\mathbb{S}^1) \oplus C^\infty(\mathbb{S}^1)$ when $b = 2$, where $\mathrm{Diff}^\infty(\mathbb{S}^1)$ denotes the group of orientation preserving diffeomorphisms of the circle and $\circledS$ denotes an appropriate semi-direct product between the pair. Recently, He and Yin [2], Chen and Zhou [3] established the local well-posedness of (1.1) in Besov spaces.

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https://doi.org/10.1016/j.nonrwa.2018.08.008
1468-1218/

Zhou [4], Zhang and Li [5] investigated the local well-posedness, blow-up criteria and Gevrey regularity of the solutions to (1.1) with $\sigma = 2$. When $\rho \equiv 0$, $\alpha = 0$ and $b = 2$, (1.1) reduces to a Camassa–Holm equation with fractional order inertia operator, whose geometrical interpretation and local well-posedness can be seen in [2,6,7], and if we further assume $2 \leq \sigma \in \mathbb{Z}_+$, (1.1) becomes a higher order Camassa–Holm equation derived as the Euler–Poincaré differential equation on the Bott–Virasoro group with respect to the H^σ metric [8].

For $\sigma = 1$, (1.1) reduces to the following nonlinear system [9]

$$\begin{cases} m_t = \alpha u_x - b u_x m - u m_x - \kappa \rho \rho_x, \quad m = u - u_{xx}, \\ \rho_t = -u \rho_x - (b-1) u_x \rho, \quad b \in \mathbb{R} \setminus \{1\}, \end{cases} \tag{1.2}$$

which models the two-component shallow water waves with constant vorticity α. In [9], Escher et al. showed the local well-posedness of (1.2) under a geometrical framework, and studied the blow-up scenarios and global strong solutions of (1.2) on the circle. In [10], Guan et al. considered the Cauchy problem of (1.2) in the Besov space and showed that the solutions have exponential decay if the initial data has exponential decay. When $\alpha = 0$, $b = 2$ and $\kappa = \pm 1$, (1.2) becomes the two-component Camassa–Holm system, which admits Lax pair and bi-Hamiltonian structure, and thus is completely integrable [11]. When $\rho \equiv 0$ and $\alpha = 0$, (1.2) reduces to a family of equations parameterized by $b \neq 1$, the so-called b-family equation. In particular, when $b = 2$ and $b = 3$, the b-family equation respectively becomes the famous completely integrable Camassa–Holm equation [12] and Degasperis–Procesi equation [13], which were introduced to model the unidirectional propagation of shallow water waves over a flat bottom. The Cauchy problem for these equations have been well-studied both on the real line and on the circle, including the well-posedness, blow-up behavior, global existence, traveling wave solutions and so on, e.g. [14–32] and the references therein.

From the results in [5], we merely know that the solution to system (1.1) with $\sigma = 2$ depends continuously on the initial data in Sobolev spaces and more general Besov spaces, but no more information about the continuity of the data-to-solution map was provided. In this work it is shown that the data-to-solution map for system (1.1) with $\sigma = 2$ is Hölder continuous in $H^s(\mathbb{R}) \times H^{s-2}(\mathbb{R})$, $s > \frac{7}{2}$, equipped with the $H^r(\mathbb{R}) \times H^{r-2}(\mathbb{R})$-topology for $1 \leq r < s$, and the Hölder exponent is expressed in terms of s and r. We mention that Hölder continuity for the b-equation was proved on the line by Chen, Liu and Zhang in [33], and for other equations were showed in [34–37]. To obtain the desired result, we need to extend the estimate of $\|fg\|_{H^{r-1}(\mathbb{R})}$ for $0 \leq r \leq 1$ in [34], commonly used in the previous works, to that of $\|fg\|_{H^{r-k}(\mathbb{R})}$ for $0 \leq r \leq k$ and $k > 1$, which plays a key role in proving the main result.

The rest of the paper is organized as follows. In Section 2, the local well-posedness for (1.1) with $\sigma = 2$ and initial data in $H^s(\mathbb{R}) \times H^{s-2}(\mathbb{R})$, $s > \frac{7}{2}$, is established, an explicit lower bound for the maximal existence time T and an estimate of the solution size are provided. The Hölder continuity of the data-to-solution map is showed in Section 3.

Throughout the paper, we denote by $\|\cdot\|_X$ the norm of Banach space X, $(\cdot,\cdot)$ the inner product of Hilbert space $L^2(\mathbb{R})$, and " $\lesssim$ " the inequality up to a positive constant.

2. Local well-posedness and estimate of the solution size

In this section, we will give the local well-posedness for Eq. (1.1) with $\sigma = 2$, and provide an explicit lower bound for the maximal existence time and an estimate of the solution size.

Setting $\Lambda^{-4} := (1 - \partial_x^2)^{-2}$, the initial-value problem associated to Eq. (1.1) with $\sigma = 2$ can be rewritten in the following form:

$$\begin{cases} u_t + u u_x + \partial_x \Lambda^{-4} \left(\frac{b}{2} u^2 + (3-b) u_x^2 - \frac{b+5}{2} u_{xx}^2 + (b-5) u_x u_{xxx} + \frac{\kappa}{2} \rho^2 - \alpha u \right) = 0, \\ \quad t > 0, \ x \in \mathbb{R}, \\ \rho_t + u \rho_x + (b-1) u_x \rho = 0, \quad t > 0, \ x \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad \rho(0,x) = \rho_0(x) \quad x \in \mathbb{R}, \end{cases} \tag{2.1}$$