Nonlinear Analysis: Real World Applications 45 (2019) 877–894

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

Local well-posedness of the complex Ginzburg–Landau equation in bounded domains

Takanori Kuroda [a,*], Mitsuharu Ôtani [b]

[a] *Major in Pure and Applied Physics, Graduate School of Advanced Science and Engineering, Waseda University, 3-4-1 Okubo Shinjuku-ku, Tokyo, 169-8555, Japan*
[b] *Department of Applied Physics, School of Science and Engineering, Waseda University, 3-4-1 Okubo Shinjuku-ku, Tokyo, 169-8555, Japan*

ARTICLE INFO

Article history:
Received 18 July 2018
Accepted 6 August 2018
Available online xxxx

Dedicated to the memory of the late Professor Kûya Masuda

Keywords:
Initial–boundary value problem
Local well-posedness
Complex Ginzburg–Landau equation
Subdifferential operator

ABSTRACT

In this paper, we are concerned with the local well-posedness of the initial–boundary value problem for complex Ginzburg–Landau (CGL) equations in bounded domains. There are many studies for the case where the real part of its nonlinear term plays as dissipation. This dissipative case is intensively studied and it is shown that (CGL) admits a global solution when parameters appearing in (CGL) belong to the so-called CGL-region. This paper deals with the non-dissipative case. We regard (CGL) as a parabolic equation perturbed by monotone and non-monotone perturbations and follows the basic strategy developed in Ôtani (1982) to show the local well-posedness of (CGL) and the existence of small global solutions provided that the nonlinearity is the Sobolev subcritical.

1. Introduction

In this paper, we consider the following Cauchy problem for the complex Ginzburg–Landau equation.

$$
\begin{cases}
\dfrac{\partial u}{\partial t}(t,x)-(\lambda+i\alpha)\Delta u-(\kappa+i\beta)|u|^{q-2}u-\gamma u=f(t,x), & \\
 & \text{in } (t,x)\in[0,T]\times\Omega,\\
u(t,x)=0 & \text{on } (t,x)\in[0,T]\times\partial\Omega,\\
u(0,x)=u_0(x) & \text{in } x\in\Omega,
\end{cases}
\tag{CGL}
$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$; λ,κ are positive parameters, α,β,γ are real parameters and $i=\sqrt{-1}$ denotes the imaginary unit; $q\geq 2$ is a given number; $f:[0,T]\to\mathbb{C}$ is a given external force defined on an interval $[0,T]$ with $T>0$. Our unknown function $u:[0,T]\to\mathbb{C}$, which takes values in the complex numbers, represents an order parameter.

* Corresponding author.
E-mail addresses: 1d_est_quod_est@ruri.waseda.jp (T. Kuroda), otani@waseda.jp (M. Ôtani).

https://doi.org/10.1016/j.nonrwa.2018.08.006

This equation is originally introduced by Ginzburg & Landau [1] in order to give a mathematical model of superconductivities. In their theory, the Ginzburg–Landau theory for superconductivities, the physical quantity u is only a parameter which describes the randomness of a certain physical state. In the sequel, the theory has been applied to more general phenomena related to phase change (Nishiura [2]) or pattern formulation (Cross & Hohenberg [3]).

There have been many contributions to the case where $\kappa < 0$: Temam [4] showed the existence of a unique weak solution using Galerkin method for the case where $N = 1, 2$ and $q = 4$; Yang [5] got mild solutions in terms of the semi-group $\left\{e^{(\lambda+i\alpha)t\Delta}; t \geq 0\right\}$ for the case where $N = 1, 2, 3$ and $q \leq 2 + \frac{4}{N}$; Okazawa & Yokota [6] developed the maximal monotone operator theory in Hilbert spaces over the complex field and proved that strong solutions exist in bounded Ω for the case where $\left(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa}\right)$ belongs to the so-called "CGL-region" and $u_0 \in \mathrm{H}^1_0 \cap \mathrm{L}^q$ as well as the case where $|\beta|/\kappa \leq 2\sqrt{q-1}/(q-2)$ and $u_0 \in \mathrm{L}^2$; in the sequel, Kuroda, Ôtani & Shimizu [7] improved this result by showing the existence of global solutions in general domains with $\left(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa}\right)$ being in the "CGL-region" and $u_0 \in \mathrm{L}^2$. In [7], the abstract theory of parabolic equations is used, where $-\lambda\Delta u$ is regarded as its leading term, $-i\alpha\Delta u$ as a monotone perturbation and $-i\beta|u|^{q-2}u$ as a non-monotone perturbation. We remark that when $\kappa < 0$ the real part of our nonlinear term $-\kappa|u|^{q-2}u$ becomes maximal monotone in L^2 and $-\lambda\Delta u - \kappa|u|^{q-2}u$ can be represented as a subdifferential operator.

On the other hand, there are few treatment for the case where $\kappa > 0$ especially on the well-posedness: Cazenave et al. studied blow-up of solutions [8,9] and the existence of standing wave solutions [10]. In their papers [8,9] they made restriction on parameters $\lambda, \kappa, \alpha, \beta$ to be $\frac{\alpha}{\lambda} = \frac{\beta}{\kappa}$. They proved the existence of a unique local solution based on the semi-group theory for sufficiently smooth initial data in the whole space $\mathbb{R}^N$ for any $q > 2$. The local well-posedness of (CGL) for general coefficients was treated in Shimotsuma, Yokota & Yoshii [11]. They showed the local existence of a unique solution in various kind of domains using the semi-group theory in L^p over the complex numbers under the assumption $2 < q < 2 + \frac{2p}{N}$. They also deduced the global extension of solutions by assuming $\frac{|\alpha|}{\lambda} < \frac{2\sqrt{p-1}}{p-2}$ and a suitable condition on γ.

The main purpose of this paper is to discuss the local well-posedness of (CGL) in a bounded domain Ω when u_0 belongs to $\mathrm{H}^1_0(\Omega)$ and q is the Sobolev subcritical, i.e.,

$$2 \leq q < 2^* := \begin{cases} +\infty & N = 1, 2, \\ \frac{2N}{N-2} & N \geq 3. \end{cases}$$

To expect the local well-posedness of (CGL) under this situation is quite natural on the analogy of the theory of nonlinear parabolic equations. In order to show that this conjecture holds true, we follow the basic strategy in [7], i.e., we regard (CGL) as a parabolic equation with the principal part $-\lambda\Delta u$ perturbed by the monotone perturbation $-i\alpha\Delta u$ and the non-monotone perturbations $-(\kappa + i\beta)|u|^{q-2}u$. To cope with these perturbations, as for the monotone perturbation, we can use the standard argument from the maximal monotone operator theory. As for the non-monotone perturbations, we rely on the Schauder–Tychonoff fixed point theory as in [12].

The outline of this paper is as follows. In Section 2, we fix function spaces over the real numbers which are direct products of usual Lebesgue or Sobolev spaces over the real numbers. Introducing suitable functionals and their subdifferentials on this space, we rewrite (CGL) in terms of an evolution equation in a real Hilbert space governed by subdifferential operators with perturbations. We conclude this section by stating our main results on the local well-posedness (Theorem 1), an alternative on the maximal existence time of solutions (Theorem 2) and the existence of small global solutions (Theorem 3).

In Section 3, we discuss the solvability of some auxiliary equations (Proposition 4), which is needed for the application of the Schauder–Tychonoff fixed point theorem. In Section 4, we establish some a priori estimates for the solutions of auxiliary equations, by which we prove the existence part of Theorem 1. Theorem 2 will be proved in Section 5 and the uniqueness part of Theorem 1 is shown in Section 6. The last section is devoted to the proof of Theorem 3.