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# Global generalized solutions to a Keller–Segel system with nonlinear diffusion and singular sensitivity

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ABSTRACT

We consider the chemotaxis system with nonlinear diffusion and singular sensitivity

$$\begin{cases} u_t = \Delta u^m - \nabla(\frac{u}{v}\nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - uv, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega \end{cases}$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$. In this work it is shown that for all reasonably regular initial data $u_0 \geq 0$ and $v_0 > 0$, the corresponding Neumann initial–boundary value problem possesses a global generalized solution provided that $m > 1 + \frac{n-2}{2n}$.



## 1. Introduction

In this paper, we consider the following chemotaxis system with nonlinear diffusion and singular sensitivity

$$\begin{cases} u_t = \Delta u^m - \nabla(\frac{u}{v}\nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - uv, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded domain with smooth boundary, $u = u(x,t)$ denotes the density of the cells, $v = v(x,t)$ represents the concentration of the oxygen and the initial data $u_0$ and $v_0$ are assumed to be nonnegative functions.

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Chemotaxis refers to the effect of chemical substances in the environment on the movement of mobile species. This can lead to strict directional movement or partial orientation and partial tumbling movement. The movement to higher chemical concentrations is called positive chemotaxis, and the movement to lower chemical concentrations is called negative chemotaxis. Chemotaxis is an important means of cellular communication. Communication through chemical signals determines how cells arrange and organize themselves, such as in development or living tissue [5].

System (1.1) is a variant of the well-known chemotaxis model initially introduced by Keller and Segel [8,9] to describe the bacterial wave propagation. When $m = 1$, (1.1) is a standard Keller–Segel system

$$\begin{cases} u_t = \Delta u - \chi\nabla(\dfrac{u}{v}\nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - uv, & x \in \Omega,\ t > 0. \end{cases} \tag{1.2}$$

The second equation models the consumption of the signal. In the first equation, the chemotactic sensitivity is determined according to the Weber–Fechner law, which says that the chemotactic sensitivity is proportional to the reciprocal of signal density. The main feature of this particular type of absorption–taxis interplay is to produce wave-like solution, and the existence and stability of traveling wave solutions have been studied in [6,14,15]. For some closely related systems, one can refer [7,12,18]. The singular chemotactic sensitivity is also very important for the system (1.2) in the modeling method, tumor angiogenesis and taxis-driven morphogen transport, see for example [1,13,16,17,19,20].

Recently, Winkler [26] showed that (1.2) possesses a global generalized solution for all reasonably regular initial data $u_0 \geq 0$ and $v_0 > 0$. The derivation of this result is based on a priori estimates for the quantities $\nabla \ln(u+1)$ and $\nabla v$ in spatio-temporal $L^2$ spaces. In higher-dimensional domains, in [28] the same author constructed renormalized solutions in a radially symmetric setting. When the term $\Delta u^m$ is replaced by $\nabla \cdot (D(u)\nabla u)$ in the first equation in (1.1), Lankeit [10] showed the existence of locally bounded global solutions of the system with $D(u) \geq \delta u^{m-1}$, $\delta > 0$, under the condition $m > 1 + \frac{n}{4}$ $(n \geq 2)$.

In contrast to system (1.1), in the related chemotaxis system

$$\begin{cases} u_t = \Delta u - \chi\nabla(\dfrac{u}{v}\nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases} \tag{1.3}$$

$v$ does not represent a nutrient to be consumed but rather a signal substance produced by the bacteria themselves. The evolution determined by the singularity in the sensitivity function can be mitigated by $v$ avoiding 0 thanks to the production term in the second equation. Winkler [24] proved that if $0 < \chi < \sqrt{\frac{2}{n}}$, there exists a global-in-time classical solution. Furthermore, relaxing the solution concept, the global existence of weak solutions is established whenever $0 < \chi < \sqrt{\frac{n+2}{3n-4}}$. In [21], Stinner and Winkler introduced a generalized solution concept, and then proved that such generalized solution exists regardless of the size of $\chi > 0$. In [11], Lankeit and Winkler introduced another generalized solution concept, which is shown to exist for the following range of $\chi$

$$\chi < \begin{cases} \infty, & \text{if } n = 2; \\ \sqrt{8}, & \text{if } n = 3; \\ \dfrac{n}{n-2}, & \text{if } n \geq 4. \end{cases}$$

The sensitivity function in the first equation of (1.3) can be replaced by more general function $\chi(v)$. For $\chi(v) = \frac{\chi_0}{(1+\alpha v)^2}$, $v > 0$, it is proved in [23] that for non-negative initial data $u(\cdot,0) \in C^0(\bar{\Omega})$ and $v(\cdot,0) \in W^{1,r}$ (with some $r > n$), the corresponding initial–boundary value problem possesses a unique global solution which is uniformly bounded. In [4], the authors proved the global existence and boundedness of classical solutions to the system with the strongly singular sensitivity function $\chi(v)$ such that $0 < \chi(v) < \frac{\chi_0}{v^k}$ $(\chi_0 > 0, k > 1)$.

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