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On Eulerian orientations of even-degree hypercubes

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ABSTRACT

It is well known that *every* Eulerian orientation of an Eulerian $2k$-edge connected (undirected) graph is strongly k-edge connected. A long-standing goal in the area is to obtain analogous results for other types of connectivity, such as node connectivity. We show that *every* Eulerian orientation of the hypercube of degree $2k$ is strongly k-node connected.

1. Introduction

The hypercube Q_k is a k-regular graph on 2^k nodes that can be constructed by labeling the nodes by the 2^k subsets of the set $\{1, 2, \ldots, k\}$ and placing an edge between two nodes whenever the two node labels (i.e., the two subsets) differ in a single element. Hypercubes (and their variants) are useful in computer communication networks, VLSI design, etc., and there is extensive literature in this area, see [3,5,8,13,15].

An *orientation* of an (undirected) graph $G = (V, E)$ is a directed graph $D = (V, A)$ such that each edge $\{v, w\} \in E$ is replaced by exactly one of the arcs (v, w) or (w, v).

Orientations of hypercubes have applications in practical domains such as broadcasting in computer communication networks and the design of parallel computer architectures. The connectivity properties of hypercubes and orientations of hypercubes have been studied, see [3,8,15], and orientations of hypercubes that achieve the maximum possible node connectivity are of interest, see [8, Proposition 9].

Our key result states that the optimal node connectivity among orientations of Q_{2k} can be achieved in a trivial way: pick any orientation such that the indegree is equal to the outdegree at every node.

1.1. Smooth orientations and Eulerian orientations

For a node v of a directed graph, we use $d^{in}(v)$ to denote the number of arcs with head v; similarly, $d^{out}(v)$ denotes the number of arcs with tail v.

An orientation of an (undirected) graph G is called *smooth* if the absolute value of the difference between the indegree and the outdegree of every node is at most one, that is, $\left|d^{in}(v) - d^{out}(v)\right| \leq 1, \ \forall v \in V(G)$. A smooth orientation of an Eulerian graph G is called an *Eulerian* orientation; such an orientation satisfies $d^{in}(v) = d^{out}(v), \ \forall v \in V(G)$. Moreover, it can be seen that for every Eulerian orientation, for every subset of the nodes W, the number of arcs leaving W is equal to the number of arcs entering W, see [4, Ch. 6.1]. Therefore, every Eulerian orientation of a $2k$-edge connected Eulerian graph results in a directed graph that is k-edge connected. An Eulerian orientation of an Eulerian graph can be found by orienting the edges of each connected component according to an Euler tour.

1.2. Nash-Williams' results and possible extensions

A well-known result of Nash-Williams says that the edges of a k-edge connected graph can be oriented such that the resulting directed graph is $\lfloor \frac{k}{2} \rfloor$-edge connected [14], [2, Ch. 9]. A long-standing goal in the area is to extend Nash-Williams' result to obtain analogous results for other types of connectivity, such as node connectivity and element connectivity, see [6,11,12,16,17].

1.3. Our results

We show that every Eulerian orientation of the hypercube Q_{2k} is strongly k-node connected; recall that a directed graph is called *strongly k-node connected* if it has $\geq k + 1$ nodes and the deletion of any set of $\leq (k - 1)$ nodes results in a strongly-connected directed graph.

Let us mention that there are easy inductive constructions that prove that there exists a "good orientation" for a hypercube of

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even degree; we describe one such construction in Fact 1. For hypercubes of odd degree, the smoothness condition does not guarantee "good orientations"; for example, there exist smooth orientations of Q_3 that are not strongly connected.

2. Preliminaries

This section has some definitions and preliminary results. Also, see [4] for standard definitions and notation.

The hypercube Q_k is the Cartesian product of k copies of K_2, see [10]. There are other constructions of Q_k, and we describe three of them.

(i) Label 2^k nodes by k-bit binary strings, and place an edge between two nodes whenever their labels differ in exactly one bit (i.e., the Hamming distance between the two strings is one).
(ii) Label 2^k nodes by the 2^k subsets of a set with k elements, and place an edge between two nodes whenever the two node labels (i.e., the two subsets) differ in a single element.
(iii) Take two disjoint hypercubes Q_{k-1}, and place an edge between corresponding pairs of nodes in the two copies of Q_{k-1}; thus, the edges between the two copies of Q_{k-1} form a perfect matching.

By a d-hypercube we mean a hypercube of degree d.

For a node set S of a graph G, we use $N_G(S)$ to denote the set of neighbors of S, thus, $N_G(S) = \{w \in V(G) - S : \exists v \in S \text{ such that } \{v, w\} \in E(G)\}$.

Fact 1. *For each integer $k \geq 1$, there exists an Eulerian orientation of Q_{2k} that is strongly k-node connected.*

Proof. Let $k \geq 1$ be an integer. We sketch an inductive construction that gives a strongly $(k+1)$-node connected Eulerian orientation for the hypercube Q_{2k+2}. Observe that any Eulerian orientation of Q_2 (the 4-cycle) is strongly 1-connected. Assume (by induction) that Q_{2k} has a strongly k-node connected Eulerian orientation. View the $(2k+2)$-hypercube as four $2k$-hypercubes (i.e., four copies of Q_{2k}) together with 2^{2k} 4-cycles, where each of these 4-cycles C_i contains a distinct node i of the first copy of Q_{2k} as well as the image of i in each of the other three copies of Q_{2k}. By the induction hypothesis, there exists a strongly k-node connected Eulerian orientation for Q_{2k}. Fix such an orientation for each of the four copies of Q_{2k}. Moreover, for each of the 4-cycles C_i, fix any Eulerian orientation of C_i. Let D be the resulting directed graph (i.e., orientation of Q_{2k+2}). We claim that D is strongly $(k+1)$-node connected. To see this, consider any set of nodes Z of size $\leq k$. Suppose that one of the four copies of Q_{2k} contains Z; then it is clear that each of the other three copies of Q_{2k} is strongly connected in $D - Z$, and hence, (using the 2^{2k} oriented 4-cycles of D) it can be seen that $D - Z$ is strongly connected. Otherwise, each of the four copies of Q_{2k} has $\leq k-1$ nodes of Z, hence, the removal of Z from any one of the four copies of Q_{2k} results in a strongly connected directed graph; again (using the 2^{2k} oriented 4-cycles of D), it can be seen that $D - Z$ is strongly connected. ■

3. Eulerian orientations of $2k$-hypercubes

This section has our results and proofs. In this section, we assume that k is a positive integer.

Theorem 2. *Let G be a $2k$-regular $2k$-node connected graph such that for every set of nodes S with $1 \leq |S| \leq |V(G)|/2$ we have $|N_G(S)| > \min\{k^2 - 1,\ (k-1)(|S|+1)\}$. Then every Eulerian orientation of G is strongly k-node connected.*

Proof. Let D denote an arbitrary Eulerian orientation of G. (In what follows, when we refer to *the orientation* of an edge of G we mean the corresponding directed edge of D.) By way of contradiction, suppose that D is not strongly k-node connected. Then there is a node set Z of size $\leq k-1$ whose deletion from D results in a directed graph that has a partition $(S, \bar{S})$ of its node set $V(G) - Z$ such that both $S, \bar{S}$ are nonempty and the edges of $G - Z$ in this cut either are all oriented from S to $\bar{S}$ or are all oriented from $\bar{S}$ to S. We fix the notation such that $|S| \leq |\bar{S}|$. (Now, observe that $|S|$ satisfies the condition stated in the hypothesis.) Moreover, without loss of generality, we assume that the edges are oriented from S to $\bar{S}$ (the arguments are similar for the other case). Observe that $G - Z$ has $\geq |N_G(S)| - |Z|$ edges in the cut $(S, \bar{S})$. Thus, D has $\geq |N_G(S)| - |Z|$ edges oriented out from S (and into $\bar{S}$). Consider the cut $(S, \bar{S} \cup Z)$ of G, and observe that it has $\leq \min\{k|Z|,\ |S|\,|Z|\}$ edges oriented into S (and out of Z), because: (i) all such edges are incident to nodes of Z and only k of the $2k$ edges incident to a node $w \in Z$ are oriented out of w; (ii) each such edge is incident to a node $s \in S$ and a node $w \in Z$ (and each pair s, w contributes at most one such edge). Thus, the cut $(S, \bar{S} \cup Z)$ of G has $\geq |N_G(S)| - |Z| \geq |N_G(S)| - (k-1)$ edges oriented out of S and $\leq \min\{k|Z|,\ |S|\,|Z|\} \leq \min\{k(k-1),\ |S|(k-1)\}$ edges oriented into S; the hypothesis (in the theorem) implies that the former quantity is greater than the latter quantity. This is a contradiction: in an Eulerian orientation of an Eulerian graph, every cut has the same number of outgoing edges and incoming edges. ■

In the next subsection we show that hypercubes of even degree satisfy all the conditions stated in Theorem 2; this gives our main result.

3.1. Bounds for the $2k$-hypercube

The main goal of this subsection is to show that the hypercube Q_{2k} satisfies the inequalities stated in Theorem 2. Our analysis has two parts depending on the size m of the set $S \subseteq V(Q_{2k})$ (in the statement of Theorem 2); the first part (Fact 4) applies for $1 \leq m \leq k+1$ and it follows easily; the second part (Fact 5) applies for $k+2 \leq m \leq 2^{2k-1}$ and it follows by exploiting properties of the hypercube. In more detail, in the second part we show that the minimum of $|N_{Q_{2k}}(S)|$ over all sets $S \subseteq V(Q_{2k})$ of size m (where $k+2 \leq m \leq 2^{2k-1}$) is $> k^2 - 1$; our proof avoids elaborate computations by exploiting structural properties of hypercubes; a key point is to focus on a subgraph of the hypercube induced by the set of binary strings of Hamming weight i and the set of binary strings of Hamming weight $i-1$ (see Claim 6 in the proof of Fact 5).

We follow the notation of [1] and use $b_v(m, Q_{2k})$ to denote $\min\{|N_{Q_{2k}}(S)| : S \subseteq V(Q_{2k}),\ |S| = m\}$; thus, $b_v(m, Q_{2k})$ denotes the minimum over all node sets $S \subseteq V(Q_{2k})$ of size m of the number of neighbors of S. For the sake of exposition, we mention that the node sets S with $|N_{Q_{2k}}(S)| = b_v(m, Q_{2k})$ (i.e., the minimizers of $b_v(m, Q_{2k})$) are Hamming balls (see [1, page 126]), and the formula for $b_v(m, Q_{2k})$ (stated in Theorem 3) is obtained by computing the minimum number of neighbors of such sets. Harper, see [9] and also see [7], proved the following result:

Theorem 3 (*Theorem 4, Ch. 16, [1]*). *Every integer m, $1 \leq m \leq 2^{2k} - 1$, has a unique representation in the form*

$$m = \sum_{i=r+1}^{2k} \binom{2k}{i} + m', \quad 0 < m' \leq \binom{2k}{r},$$

$$m' = \sum_{j=s}^{r} \binom{m_j}{j}, \quad 1 \leq s \leq m_s < m_{s+1} < \cdots < m_r.$$

Moreover,

$$b_v(m, Q_{2k}) = \binom{2k}{r} - m' + \sum_{j=s}^{r} \binom{m_j}{j-1}.$$