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Spatial synchrony in fractional order metapopulation cholera transmission

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ABSTRACT

Movement of individuals within metapopulations is characterised by individuals frequenting their home ranges. This not only constitutes memory but also nonlocal property of the resulting system making it plausible to be modelled by Fractional order differential equations. In this paper, we propose a fractional order metapopulation model for transmission of cholera between communities with differing standards of living. Important basic properties of the model such as non-negativity of solutions as well as boundedness are tested. The solutions to the model are shown to exist and the steady state is unique whenever it exists. The model is numerically integrated using the iterative Adams-Bashforth-Mouton method. Our results show that, there is increase synchronous fluctuation in the population of infected individuals in connected communities with either restricted movement or with unrestricted movement of susceptible and infected individuals. In communities with movement restricted to only susceptible individuals, synchronous fluctuation of the infected population in the two communities is more pronounced at lower orders of the fractional derivatives. In unrestricted communities however, the infected population in the two adjacent communities synchronously regardless of the order of the fractional derivative.

1. Introduction

Cholera is a gastroenteritic infection caused by a pathogen Vibrio cholerae [1]. The mode of transmission is characterised by two pathways; the primary route where the epidemiologically naive individual consumes the pathogen from vibrio contaminated water; the secondary route which is also characterised by epidemiologically naive individuals consuming Vibrio cholerae from contaminated food from cholera patients or carriers. This secondary route of transmission is also commonly referred to as person-to-person contact [1]. Cholera infection has affected almost all parts of the world. Its devastating force has however been more pronounced in relatively impoverished communities. The associated symptoms and possible control measure of cholerae have been highlighted by many researchers, see for instance [1–5] among many others. It can be noted that Cholera is one of the most extensively studied infections in the recent years. Mathematical models have been used to study and understand the dynamics of the infection as well as offer possible suggestions toward its control, see for instance[1–9]. Majority of the models used in the study of cholera have been mainly based on systems of integer ordinary differential equations (IOODEs). Such models do not fully account for memory as well as nonlocal properties exhibited by the epidemic system being studied. Owing to such properties, it is plausible to model the epidemic's dynamics using Fractional order differential equations.

Integer order ordinary differential equations (IOODE) models have been widely used to model the transmission dynamics of infectious diseases as well as the dynamics of various process in biological systems. Modelling with IOODEs mainly accounts for the time evolution of the system or the infections without full regard of the after-effects or memory that is often exhibited in most biological systems and infections in general. Therefore, if after-effects and memory exhibited in biological systems and infectious diseases transmission dynamics are to be accounted for, Fractional Order Differential Equations (FODEs) ought to be used instead of the classical IOODEs. Fractional order diffential equations have been used to model the dynamics of infections namely, hepatitis B virus [10], HIV [11,12], Hepatitis C [13], Dengue fever [14], chaotic dynamics in cancer modelling [15] among others. Other recent studies involving FODEs include; synchronisation of circadian rhythms [16], bioluminescence behaviour [17], Chaotic behaviour [18,19], hyperchaotic behaviour [20], chemotaxis modelling [21], anomalous diffusion [22], evolution equations of fraction order [23], magnetic field effects and thermal radiation in oscillatory arteries [24],

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Nonlinear Baggs–Freedman Model [25], FitzHugh–Nagumo oscillations [26], and capturing more natural phenomena by breaking commutativity and associativity [27].

In the recently study of cholera transmission in adjacent communities [4], a system of IOODE was used. It was indicated that individuals in adjacent communities often frequent their home range, an aspect which constitutes memory. The justification for a metapopulation cholera study was based on the 2000–2002 cholera epidemic that devastated mainly the KwaZulu Natal (KZN) province of South Africa. Although the infection spread to eight of the nine provinces of the Republic of South Africa, more that 95% of the cases were confined within KZN province (see[4] and the references there in). During cholera outbreaks, rarely does the infection concentrate in only one locality. Most commonly, the infection in affected metapopulations is driven by subpopulations primarily consuming the pathogen from a common water source that could be traversing through the communities under consideration. However, this is not always the case if we consider the 2000–2002 cholera epidemic where the eight provinces affected are not all connected by a single common water source. Since cholera is also known to be transmitted via a secondary route through consumption of *Vibrio cholerae* foods [1], the secondary route of transmission can not be ignored.

Movement of individuals within metapopulations is characterised by individuals frequenting their home range [28]. This is typical of a resulting process exhibiting memory, thus making it plausible to model such a system using FODEs. Contrary to integer order differential operators, FODEs models exhibit nonlocal behaviour, a property which asserts that the subsequent state of the model depends on both the current and historical states of such a model.

This paper is organised as follows; in Section 2 a model formulation is presented, definitions of some types of fractional order differential equations are given and basic properties of the model including positivity and boundedness of solutions highlighted; in Section 3, the existence and uniqueness of solutions for the model considered is proved following the fixed point theory approach used by Ullah et al. [10]; the iterative numerical procedure used to simulate the model as well as numerical results are given in Sections 4 and 5 a conclusion is presented.

2. Mathematical model

Njagarah and Nyabadza [4] developed a metapopulation model for cholera transmission assuming that individuals infected with cholera recover from the infection with acquired immunity that wanes over time. The metapopulation model was based on a system of integer order ordinary differential equations with the adjacent communities assumed to contract the infection through consumption of Vibrio cholerae contaminated water from unconnected water sources as well as consumption of contaminated foods but particularly localised within their home communities. The interconnection of the communities under consideration is through movement of either susceptible or infected individuals across communities with an assumption that individuals follow the dynamics of destination community. Recovered individuals were assumed to be less likely to move across communities having learnt from their past experiences and due to fear of reinfection. Owing to the memory aspect assumed to be exhibited, the metapopulation model is studied using FODEs in this paper to ascertain vital dynamics that might have not been captured by the integer order differential equations model type presented in [4]. Several approaches for generalisation of fractional order differentiation have been proposed, for example, Riemann–Liouville, Caputo-fractional derivative as well as Generalised functions approaches among others. For convenience of the reader, we give the definitions of the Riemann–Liouville, Caputo-fractional derivatives since the work presented in this paper is centred around Caputo-fractional derivative which is a modification of the Riemann–Liouville fractional derivative.

Let $L^1 = L^1[a, b]$ be a class of Lebesque integrable functions on $[a, b]$, $a < b < \infty$.

Definition 1. The fractional integral (or the Riemann–Liouville integral) of order $\alpha \in \mathbb{R}^+$ of the function $g(t), t > 0$ $(g : \mathbb{R}^+ \to \mathbb{R})$ is defined by

$$I_a^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} g(s) d(s), \quad t > 0 \tag{1}$$

The fractional derivative of order $\alpha \in (n-1, n)$ of $g(t)$ is defined in two (non equivalent) ways

(i) Riemann–Liouville fractional derivative: take fractional integral of order $(n-\alpha)$, and then take nth derivative as follows

$$D_*^\alpha g(t) = D_*^\alpha I_a^{n-\alpha} g(t), \quad D_*^n = \frac{d^n}{dt^n}, \quad n = 1, 2, 3, \ldots.$$

(ii) The generalised Caputo-fractional derivative: take nth derivative and then take a fractional integral of order $(n-\alpha)$

$$_aD^\alpha g(t) = I_a^{n-\alpha} D_*^\alpha g(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{g^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}, & n-1 < \alpha < n, \\ \frac{d^n}{dt^n} g(t) & \alpha = n, \end{cases}$$

where n is the first integer greater that α, $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ - a gamma function.

Owing to the need to consider fractional derivatives with nonlocal and non-singular kernel, other fractional differential equations have since been developed as defined below;

Definition 2. The Antagana–Baleanu (ABC) fractional derivative in Caputo sense [29] Let $f \in H^1(a, b)$, $a < b$, $\alpha \in [0, 1]$ then, the ABC fractional derivative is given as:

$$^{ABC}_{\ a}D_t^\alpha (f(t)) = \frac{B(\alpha)}{1-\alpha} \int_a^t f'(x) E_\alpha \left[-\alpha \frac{(t-x)^\alpha}{1-\alpha} \right] dx,$$

where $B(\alpha)$ has the same properties as the Caputo Fabrizio case.

Definition 3. The Antagana–Baleanu (ABR) fractional derivative in Riemann–Liouvile sense [29]: Let $f \in H^1(a, b)$, $a < b$, $\alpha \in [0, 1]$ then, the ABC fractional derivative is given as:

$$^{ABR}_{\ a}D_t^\alpha (f(t)) = \frac{B(\alpha)}{1-\alpha} \int_a^t f(x) E_\alpha \left[-\alpha \frac{(t-x)^\alpha}{1-\alpha} \right] dx.$$

Definition 4. The Caputo–Fabrizio fractional order [30] is given by

$$D_t^\alpha f(t) = \frac{\mathcal{M}(\alpha)}{(1-\alpha)} \int_a^t \dot{f}(\tau) \exp\left[-\frac{\alpha(t-\tau)}{1-\alpha} \right] d\tau$$

with $\mathcal{M}(\alpha)$ a normalised function such that $\mathcal{M}(0) = \mathcal{M}(1) = 1$. For detailed explanations about this Fractional derivative, readers are referred to Ref. [30].

Since the model considered in this paper is an initial value problem, we use the specific case of the generalised Caputo-Fractional derivative with $a = t_0 = 0$ when dealing with the basic properties of the model. Caputo's definition has the advantage of dealing appropriately with initial value problems [14], a drawback often associated with combining Riemann–Liouville differential equations with classical initial conditions. We note that the definition of time-fractional derivative of a function $g(t)$ at $t = t_n$ involves an integration and calculating time-fractional derivative that requires all the past history, i.e all values of $g(t)$ from $t = 0$