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Bar code for monomial ideals

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ABSTRACT

The aim of this paper is to count 0-dimensional stable and strongly stable ideals in 2 and 3 variables, given their (constant) affine Hilbert polynomial p, by means of a bijection between these ideals and some integer partitions of p, which can be counted via determinantal formulas. This will be achieved by the *Bar Code*, a bidimensional diagram that allows to represent any finite set of terms M and desume many properties of the corresponding monomial ideal I, if M is an order ideal.

1. Introduction

Strongly stable ideals play a special role in the study of Hilbert scheme, which was first introduced by Grothendieck (1960). Indeed, their escalier allows to study the Hilbert function of any homogeneous ideal using the theory of Groebner bases, as shown by Bayer (1983) and Eisenbud (2013).

The notion of generic initial ideal was introduced by Galligo (1974) with the name of *Grauert invariant*. Galligo proved that the generic initial ideal of any homogeneous ideal is closed w.r.t the action of the Borel group. He then gave a combinatorial characterization of such ideals, provided that they are defined over a field of characteristic zero. Eisenbud (2013) and Peeva (1996) called these monomial ideals 0-*Borel-fixed ideals*; Aramova and Herzog (1996), Aramova and Herzog (1997) renamed them *strongly stable ideals*. A combinatorial description of ideals that are closed w.r.t the action of the Borel group over a polynomial ring over a field of characteristic $p > 0$ has been provided by Pardue (1994). And Galligo's result has been extended to that setting by Bayer and Stillman (1987).

The notion of *stable ideal* has been introduced by Eliahou and Kervaire (1990) as a generalization of 0-Borel-fixed ideals. In their work, they gave a minimal resolution for stable ideals, that was then used

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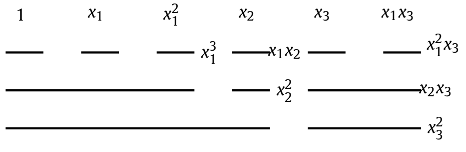
by Bigatti (1993) and Hulett (1993) to extend Macaulay's result (Macaulay, 1927); they proved that the lex-segment ideal has maximal Betti numbers, among all ideals with the same Hilbert function. In connection with the study of Hilbert schemes (see Bertone et al., 2013b,a; Cioffi and Roggero, 2011; Lella et al., 2016; Moore and Nagel, 2014; Reeves, 1993), it has been considered relevant to list all stable ideals (Bertone, 2015) and strongly stable ideals (Cioffi et al., 2011; Lella, 2012) with a fixed Hilbert polynomial.

The aim of this paper is to count zerodimensional stable and strongly stable ideals in 2 and 3 variables, given their (constant) affine Hilbert polynomial. To do so, we first introduce a bidimensional structure – the *Bar Code* – which allows, a priori, to represent any (finite[1]) set of terms M and, if M is an order ideal, to desume many combinatorial properties of the corresponding monomial ideal I (Ceria, 2018a,b; Ceria and Mora, 2018). For example, a Pommaret basis (Seiler, 2009; Ceria et al., 2015) of I can be easily desumed. The Bar Code is strictly connected to Felzeghy–Rath–Ronyay's Lex Trie (Felszeghy et al., 2006; Lundqvist, 2010), but our goal and methods are different. With the Bar Code, we provide a connection between zerodimensional (strongly) stable monomial ideals and integer partitions.

In the two-variable case, there is a bijection between (strongly) stable ideals with affine Hilbert polynomial p and partitions of p with distinct parts.

The case of three variables is more complicated and more technology is required. Thanks to the Bar Code, though, we can provide a bijection between (strongly) stable ideals and some special plane partitions of their constant affine Hilbert polynomial p. These partitions have been studied by Krattenthaler (1990, 1993), who proved determinantal formulas to find their norm generating functions and finally count them.

As an example, we take the stable monomial ideal $I_1 = (x_1^3, x_1x_2, x_2^2, x_1^2x_3, x_2x_3, x_3^2) \triangleleft \mathbf{k}[x_1, x_2, x_3]$, whose Groebner escalier is $\mathsf{N}(I_1) = \{1, x_1, x_1^2, x_2, x_3, x_1x_3\}$. We represent it by the Bar Code and the plane partition below



$$\begin{array}{cc} 3 & 1 \\ 2 & \end{array}$$

The correspondence can be seen observing the rows of the Bar Code above: since the bottom row is composed by two segments, the plane partition has exactly two rows. The number of entries in the i-th row of the partition, $i = 1, 2$ (i.e. 2 and 1 resp.), is given by the number of segments in the middle-row, lying over the i-th segment of the bottom row. Finally, the entries are represented by the number of segments in the top row, lying over the segments representing the corresponding entry. With this bijection and the determinantal formulas by Krattenthaler, we are able to count (strongly) stable ideals in three variables. A Bar Code can in principle represent finite sets of terms in any number of variables. Nevertheless, we do not generalize our results to the case of 4 or more variables because it would require the introduction of n-dimensional partitions. And – in my knowledge – the way to count them.[2]

2. Some algebraic notation

Throughout this paper we mainly follow the notation of Mora (2005), for what concerns monomial ideals. We denote by $\mathcal{P} := \mathbf{k}[x_1, \dots, x_n]$ the graded ring of polynomials in n variables with coefficients in the field $\mathbf{k}$.

[1] There is also the possibility to have *infinite* Bar Codes for infinite sets of terms, but it is out of the purpose of this paper, so we will only see an example for completeness' sake.

[2] In Andrews (1998), Chapter 11, the author observes that "*Surprisingly, there is much of interest when the dimension is* 1 *or* 2, *and very little when the dimension exceeds* 2."