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Computing the monodromy and pole order filtration on Milnor fiber cohomology of plane curves

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ABSTRACT

We describe an algorithm computing the monodromy and the pole order filtration on the Milnor fiber cohomology of any reduced projective plane curve C. The relation to the zero set of Bernstein–Sato polynomial of the defining homogeneous polynomial for C is also discussed. When C has some non-weighted homogeneous singularities, then we have to assume that a conjecture holds in order to get some of our results. In all the examples computed so far this conjecture holds.

1. Introduction

Let $C : f = 0$ be a reduced plane curve of degree $d \geq 3$ in the complex projective plane $\mathbb{P}^2$, defined by a homogeneous polynomial $f \in S = \mathbb{C}[x, y, z]$. Consider the corresponding complement $U = \mathbb{P}^2 \setminus C$, and the global Milnor fiber F defined by $f(x, y, z) = 1$ in $\mathbb{C}^3$ with monodromy action $h : F \to F$,

$$h(x, y, z) = \exp(2\pi i/d) \cdot (x, y, z).$$

To determine the eigenvalues of the monodromy operators

$$h^m : H^m(F, \mathbb{C}) \to H^m(F, \mathbb{C}) \tag{1.1}$$

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for $m = 1, 2$ starting from C or f is a rather difficult problem, going back to O. Zariski and attracting an extensive literature, see for instance Artal Bartolo (1994), Artal Bartolo and Dimca (2015), Budur et al. (2011), Cohen and Suciu (1995), Dimca (1992, 2017), Esnault (1982), Libgober (1982, 2011), Oka (2005), Degtyarev (2012), Suciu (2014). When the curve $C : f = 0$ is either free or nearly free, we have presented in Dimca and Sticlaru (2017a) an efficient algorithm for listing the eigenvalues of the monodromy operator h^1, which determines completely the corresponding Alexander polynomial $\Delta_C(t)$, see Remark 4.4 below for its definition.

In this paper we explain an approach working in the general case. This time the results of our computation give not only the dimensions of the eigenspaces $H^m(F, \mathbb{C})_\lambda$ of the monodromy, but also the dimensions of the graded pieces $Gr_P^p H^m(F, \mathbb{C})_\lambda$, where P denotes the pole order filtration on $H^m(F, \mathbb{C})$, see section 2 below for the definition. More precisely, the algorithm described here gives the following.

(1) the dimensions of the eigenspaces $H^m(F, \mathbb{C})_\lambda$ for $m = 1, 2$ for any reduced curve $C : f = 0$, see Remark 4.4.
(2) the dimensions of the graded pieces $Gr_P^p H^1(F, \mathbb{C})_\lambda$, for any reduced curve $C : f = 0$. Note that the P^p filtration coincides to the Hodge filtration F^p on $H^1(F, \mathbb{C})$, see Dimca and Sticlaru (2017b, Proposition 2.2).
(3) the dimensions of the graded pieces $Gr_P^p H^2(F, \mathbb{C})_\lambda$, for a reduced curve $C : f = 0$ having only weighted homogeneous singularities. To achieve this efficiently one has to use the recent result by M. Saito stated below in Theorem 1.1.
(4) the dimensions of the graded pieces $Gr_P^p H^2(F, \mathbb{C})_\lambda$, for any reduced curve $C : f = 0$ under the assumption that a basic fact, stated as Conjecture 2.6, holds. This conjecture holds in all the examples we have computed so far, see Remark 6.2.

The new information on the pole order filtration P can be applied to describe the set of roots of $b_f(-s)$, where $b_f(s)$ is the Bernstein–Sato polynomial of f, see for details Saito (2007, 2015). In fact, using Saito (2007, Theorem 2), this comes down to checking whether $Gr_P^p H^2(F, \mathbb{C})_\lambda \neq 0$, see Theorem 7.2 for a precise statement and our applications described in Corollaries 7.3, 7.4, 7.5.

Here is in short how we proceed. Let Ω^j denote the graded S-module of (polynomial) differential j-forms on $\mathbb{C}^3$, for $0 \leq j \leq 3$. The complex $K_f^* = (\Omega^*, \mathrm{d} f \wedge)$ is just the Koszul complex in S of the partial derivatives f_x, f_y and f_z of the polynomial f. The general theory says that there is a spectral sequence $E_*(f)$, whose first term $E_1(f)$ is computable from the cohomology of the Koszul complex K_f^* and whose limit $E_\infty(f)$ gives us the action of monodromy operator on the graded pieces of the reduced cohomology $\tilde{H}^*(F, \mathbb{C})$ of F with respect to the pole order filtration P, see Dimca (1990), Dimca (1992, Chapter 6), Dimca and Saito (2014).

Our approach takes a simpler form when C is assumed to have only weighted homogeneous singularities, e.g. when C is a line arrangement $\mathcal{A}$. This comes from the following result due to Saito (2017a), see for a more precise statement Theorem 2.1 below.

Theorem 1.1. *If the reduced plane curve $C : f = 0$ has only weighted homogeneous singularities, then the spectral sequence $E_*(f)$ degenerates at the E_2-term.*

This result has been conjectured already in Dimca (1990) and has been checked in many cases using a computer program in Dimca and Sticlaru (2017a). The converse implication is known to hold, see Dimca and Saito (2014, Theorem 5.2). The algorithm described in Dimca and Sticlaru (2017a) for free and nearly free curves actually computes the E_2-term of this spectral sequence.

In this paper we modify the algorithm in Dimca and Sticlaru (2017a) such that it applies to any reduced curve. First we compute (a large part) of the E_2-term in Section 4, which is (more than) enough when C has only weighted homogeneous singularities, see Remark 4.3, or when we are interested only in the Alexander polynomial $\Delta_C(t)$. Then in Section 5 we compute the relevant part of the E_3-term of the above spectral sequence. Conjecture 2.6 tells us that essentially $E_3 = E_\infty$ and that we

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