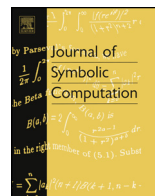




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# Algorithms for tight spans and tropical linear spaces

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## ABSTRACT

We describe a new method for computing tropical linear spaces and more general duals of polyhedral subdivisions. This is based on an algorithm of Ganter for finite closure systems.

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## 1. Introduction

Tropical linear spaces are among the most basic objects in tropical geometry (Maclagan and Sturmfels, 2015, Chapter 4). In combinatorial terms they form polyhedral complexes which are dual to regular matroid subdivisions of hypersimplices. Such subdivisions are characterized by the property

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that each cell is the convex hull of characteristic vectors of the bases of a matroid. Here the hyper-simplices correspond to the uniform matroids. Research on matroid subdivisions and related objects goes back to Dress and Wenzel (1992) and to Kapranov (1993). Speyer instigated a systematic study in the context of tropical geometry (Speyer, 2008), while suitable algorithms have been developed and implemented by Rincón (2013).

Here we present a new combinatorial algorithm for computing tropical linear spaces, which are not necessarily realizable. This is implemented in the software system `polymake` (Gawrilow and Joswig, 2000). Moreover, we report on computational experiments. Our approach has two key ingredients. First, our method is completely polyhedral – in contrast with Rincón's algorithm (Rincón, 2013), which primarily rests on exploiting matroid data. Employing the polyhedral structure has the advantage that this procedure naturally lends itself to interesting generalizations and variations. In particular, this includes tropical linear spaces corresponding to non-trivially valuated matroids. Second, our method fundamentally relies on an algorithm of Ganter (1987) for enumerating the closed sets in a finite closure system; cf. Ganter and Obiedkov (2016, §2.2). This procedure is a variant of breadth-first-search in the Hasse diagram of the poset of closed sets. As a consequence the computational costs grow only linearly with the number of edges in the Hasse diagram, i.e., the number of covering pairs among the closed sets. So this complexity is asymptotically optimal in the size of the output, and this is what makes our algorithm highly competitive in practice. The challenge is to implement the closure operator and to intertwine it with the search in such a way that it does not impair the output-sensitivity.

Kaibel and Pfetsch employed Ganter's algorithm for enumerating face lattices of convex polytopes (Kaibel and Pfetsch, 2002), and this was later extended to bounded subcomplexes of unbounded polyhedra (Herrmann et al., 2013). Here this is generalized further to arbitrary regular subdivisions and their duals. Such a dual has been called *tight span* in Herrmann et al. (2012) as it generalizes the tight spans of finite metric spaces studied by Isbell (1964) and Dress (1984). The tight span of an arbitrary polytopal complex may be seen as a special case of the dual block complex of a cell complex; e.g., see Munkres (1984, §64). From a topological point of view subdivisions of point configurations are cell decompositions of balls, which, in turn, are special cases of manifolds with boundary. The duality of manifolds with boundary is classically known as Lefschetz duality (e.g., see Munkres, 1984, §70), and this generalizes Poincaré duality as well as cone polarity. With an arbitrary polytopal subdivision,  $\Sigma$ , we associate a new object, called the *extended tight span* of  $\Sigma$ , which contains the tight span, but which additionally takes duals of certain boundary cells into account. In general, the extended tight span is only a partially ordered set. If, however,  $\Sigma$  is regular, then the extended tight span can be equipped with a natural polyhedral structure. We give an explicit coordinatization. In this way tropical linear spaces arise as the extended tight spans of matroid subdivisions with respect to those boundary cells which correspond to loop-free matroids. While a tropical linear space can be given several polyhedral structures, the structure as an extended tight span is the coarsest. Algorithmically, this has the advantage of being the sparsest, i.e., being the one which takes the least amount of memory. In this sense, this is the canonical polyhedral structure of a tropical linear space.

This paper is organized as follows. We start out with recalling basic facts about general closure systems with a special focus on Ganter's algorithm (Ganter, 1987). Next we introduce the extended tight spans, and this is subsequently specialized to tropical linear spaces. We compare the performance of Rincón's algorithm (Rincón, 2013) with our new method; see also Section 4.2 for further comments. To exhibit one application the paper closes with a case study on the  $f$ -vectors of tropical linear spaces.

## 2. Closure systems, lower sets and matroids

While we are mainly interested in applications to tropical geometry, it turns out that it is useful to start out with some fundamental combinatorics. This is the natural language for Ganter's procedure, which we list as Algorithm 1 below.

**Definition 2.1.** A *closure operator* on a set  $S$  is a function  $\text{cl} : \mathcal{P}(S) \rightarrow \mathcal{P}(S)$  on the power set of  $S$ , which fulfills the following axioms for all subsets  $A, B \subseteq S$ :

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