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The tropical analogue of the Helton–Nie conjecture is true ☆

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ABSTRACT

Helton and Nie conjectured that every convex semialgebraic set over the field of real numbers can be written as the projection of a spectrahedron. Recently, Scheiderer disproved this conjecture. We show, however, that the following result, which may be thought of as a tropical analogue of this conjecture, is true: over a real closed nonarchimedean field of Puiseux series, the convex semialgebraic sets and the projections of spectrahedra have precisely the same images by the nonarchimedean valuation. The proof relies on game theory methods.

1. Introduction

Convex semialgebraic sets appear in various guises in computational optimization (Blekherman et al., 2013). They include spectrahedra, i.e., feasible sets of semidefinite programs (SDPs). A long-standing problem is to characterize the convex semialgebraic sets that are SDP representable, meaning that they can be represented as the image of a spectrahedron by a (linear) projector. The notion of SDP representability originates from the monograph of Nesterov and Nemirovskii (1994).

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Nemirovski (2007) asked whether every convex semialgebraic set is SDP representable. Helton and Nie conjectured that the answer is positive.

Conjecture 1 *(Helton and Nie, 2009). Every convex semialgebraic set in $\mathbb{R}^n$ is a projection of a spectrahedron.*

Several classes of convex semialgebraic sets for which the answer is positive have been identified (Helton and Nie, 2009, 2010; Helton and Vinnikov, 2007; Lasserre, 2009; Gouveia et al., 2010; Gouveia and Netzer, 2011; Nie et al., 2008). In particular, it is known that the conjecture is true in dimension 2 (Scheiderer, 2018a). The conjecture has been recently disproved by Scheiderer (2018b), who showed that the cone of positive semidefinite forms cannot be expressed as a projection of spectrahedra, except in some particular cases. A comprehensive list of references can be found in this work.

Theorem 2 *(Scheiderer, 2018b). The cone of positive semidefinite forms of degree $2d$ in n variables can be expressed as a projection of a spectrahedron only when $2d = 2$ or $n \leqslant 2$ or $(n, 2d) = (3, 4)$.*

The notion of convex and semialgebraic sets make sense over any real closed field, in particular over the nonarchimedean field $\mathbb{K}$ of real Puiseux series, equipped with the total order induced by its nonnegative cone $\mathbb{K}_{\geqslant 0}$, consisting of series with a nonnegative leading coefficient. Our main result shows that the next statement, which may be thought of as a "Helton–Nie conjecture for valuations," is valid.

Theorem 3. *The image by the valuation of every convex semialgebraic subset of $\mathbb{K}^n$ coincides with the image by the valuation of a projected spectrahedron over $\mathbb{K}$.*

Our approach relies on tropical methods. Tropical semialgebraic sets can be defined as the images by the nonarchimedean valuation of semialgebraic sets over $\mathbb{K}$. The quantifier elimination techniques in real valued fields developed by Pas (1989), building on work of Denef (1986), imply that tropical semialgebraic sets are semilinear. Moreover, the image by the nonarchimedean valuation of a convex set over $\mathbb{K}$ is a tropical convex set, i.e., a set stable by taking tropical convex combinations. In a previous work (Allamigeon et al., 2016) we studied tropical spectrahedra, defined as the images by the nonarchimedean valuation of spectrahedra over $\mathbb{K}$, and gave a combinatorial characterization of generic tropical spectrahedra.

The proof relies on the recently developed relations between tropical convex programming and zero-sum games (Akian et al., 2012; Allamigeon et al., 2018b). In particular, in the latter reference, we demonstrated a class of generic tropical spectrahedra that corresponds precisely to the sets of subharmonic vectors (subfixed points) of a class of nonlinear Markov operators (Shapley operators of stochastic mean payoff games). In that way, one obtains an explicit construction for these tropical spectrahedra.

The tropical perspective proved to be useful to find counterexamples to classical conjectures in real algebraic geometry. For instance, Itenberg and Viro (1996) disproved the Ragsdale conjecture as an application of the tropical patchworking method. More recently, Allamigeon et al. (2018a) contradicted, by a tropical method, the continuous analogue of the Hirsch conjecture proposed by Deza et al. (2008). The validity of the tropical analogue of the Helton–Nie conjecture raises the question whether a counterexample could be found by a tropical approach, for instance, by studying images of convex semialgebraic sets and spectrahedra through a map carrying more information than the valuation.

We finally note that semilinear sets that are tropically convex have been studied recently by Bodirsky and Mamino (2016) from a different perspective, motivated by a class of satisfiability problems. They showed in particular that feasibility and infeasibility certificates for these problems can be obtained from stochastic games. The tropical convex sets they consider differ from ours in two respects: the $-\infty$ element is not allowed in their approach, whereas it appears as the image of the zero element by the nonarchimedean valuation; moreover, the tropicalizations of convex semialgebraic sets are always closed, and so, definable by weak inequalities, whereas systems including both strict and weak inequalities are considered in Bodirsky and Mamino (2016).

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