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Syntactic forcing models for coherent logic

Marc Bezem[a,*], Ulrik Buchholtz[b,c], Thierry Coquand[d]

[a] *Department of Informatics, University of Bergen, Norway*
[b] *Department of Philosophy, Carnegie Mellon University, United States*
[c] *Department of Mathematics, Technische Universität Darmstadt, Germany*
[d] *Department of Computer Science and Engineering, Chalmers/University of Gothenburg, Sweden*

Abstract

We present three syntactic forcing models for coherent logic. These are based on sites whose underlying category only depends on the signature of the coherent theory, and they do not presuppose that the logic has equality. As an application we give a coherent theory T and a sentence ψ which is T-redundant (for any geometric implication φ, possibly with equality, if $T + \psi \vdash \varphi$, then $T \vdash \varphi$), yet false in the generic model of T. This answers in the negative a question by Wraith.

1. Introduction

Coherent logic concerns implications between positive formulas, those built up from atoms using only the connectives $\top$, $\bot$, $\vee$, $\wedge$ and $\exists$. A first-order theory T is *coherent* if it is axiomatized by sentences of the form $\forall\vec{x}.\,\varphi \to \psi$ where φ, ψ are positive formulas. Such sentences are also called *coherent implications* or *coherent sentences*.[1] Any coherent implication is equivalent to a finite conjunction of sentences of the form

$$\forall\vec{x}.\,(\varphi_0 \to \exists\vec{x}_1.\varphi_1 \vee \cdots \vee \exists\vec{x}_k.\varphi_k), \tag{1}$$

where the φ_i are conjunctions of atoms, and we can and will thus always assume that any coherent theory T is presented by axioms of this form.

* Corresponding author.
E-mail addresses: bezem@ii.uib.no (M. Bezem), buchholtz@mathematik.tu-darmstadt.de (U. Buchholtz), coquand@chalmers.se (T. Coquand).

[1] NB sometimes positive formulas are also called coherent formulas, but a coherent sentence in our sense need not be a coherent formula in this sense.

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Coherent theories include all universal Horn theories. The axioms of equality are all coherent implications, and over logic with equality all algebraic theories, as well as the theories of fields and local rings, are coherent. Coherent implications form a *Glivenko class*, i.e., if a coherent implication is derivable from a coherent theory using classical logic, then it is already so derivable intuitionistically. Furthermore, they are preserved by the inverse image parts of geometric morphisms between toposes, and for every coherent theory T there is a *generic model* M_T in a sheaf topos $\mathbf{Set}[T]$, called the *classifying topos* of T, classifying models of T in any sheaf topos (cf. Section 4.1 for precise definitions). The geometric perspective offered by sheaf toposes also motivates the introduction of *geometric theories*, those axiomatized by *geometric implications*, i.e., implications between infinitary positive formulas, or equivalently, those with axioms of the form (1) where the disjunction is allowed to be infinite (as usual when dealing with infinitary fragments, we require that all formulas only have finitely many free variables). We assume the existence of a Grothendieck universe or some other means of talking about small sets. We then only consider infinitary conjunctions and disjunctions indexed by small sets.

The generic model of a coherent theory T can be thought of as a forcing model, and in this paper we present three other forcing models for coherent logic without equality, providing proofs of soundness and completeness. The models can be understood as living in certain sheaf toposes, and in a companion paper we shall characterize these as classifying toposes of certain geometric theories related to T.

A main goal of the current paper is to answer the following question of Wraith:

> *The problem of characterizing all the non-geometric properties of a generic model appears to be difficult. If the generic model of a geometric theory T satisfies a sentence α then any geometric consequence of $T + (\alpha)$ has to be a consequence of T. We might call α T-redundant. Does the generic T-model satisfy all T-redundant sentences?* [17, p. 336]

We shall answer this in the negative, even for a coherent theory T. For the current volume in honour of Brouwer it is appropriate to clarify that the question makes constructive sense, and that our answer is constructive too. The proper understanding of Wraith's question requires a fair amount of categorical logic, which we develop in Section 2. For the construction of the generic model we rely on Coste and Coste [2]. These preparations postpone the proof of the negative answer to Wraith's question until the very last section of this paper. Therefore we find it useful to sketch this proof already here.

We shall give a consistent coherent theory T and a sentence ψ such that both ψ and $\neg\psi$ are T-redundant. Since the generic model of T cannot satisfy both ψ and $\neg\psi$, this provides a negative answer to Wraith's question. However, this answer is not as informative as one would hope: one would like to know which one of ψ and $\neg\psi$ is true in the generic model of T. For this we have to take closer look at the argument why both ψ and $\neg\psi$ are T-redundant.

Soundness for our forcing models means that every intuitionistic consequence (coherent or not, possibly with $=$) of T is forced. In fact, we prove soundness for all infinitary formulas, including all geometric implications. Geometric completeness means that any (generalized) *geometric implication without* $=$ that is forced is an intuitionistic consequence of the coherent theory.

Now let ψ be any sentence without $=$ that is forced in any one of our models. Let φ be a geometric implication without $=$ such that $\psi \to \varphi$ is intuitionistically provable in T. Then by soundness $\psi \to \varphi$ is forced, and hence φ is forced. Since φ is geometric without $=$ it follows by geometric completeness that φ is provable in T *without using* ψ. A sentence like ψ is thus T-redundant in logic without equality. It can then be shown by a cut-elimination argument that ψ is also T-redundant in logic with equality.

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