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Surveyable sets

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Abstract

Call a set existentially (or universally) surveyable if existential (or universal) quantification over that set preserves decidability in the sense of intuitionistic logic. We study these notions of surveyability, their preservation properties, and their connections with each other and with the related notion of completeness of the two-element lattice.

1. Introduction

When I teach a basic logic course, I begin the discussion of (classical) sentential logic by explaining that the sentences to be considered should be clear, declarative sentences — not questions, not commands, not vague statements. They should be definitely true or false. With this convention, classical sentential logic is entirely justified; the law of the excluded middle has been built into the very notion of sentence, and all Brouwerian objections to it are blocked.

Of course, this definitional approach to the law of the excluded middle does not really refute intuitionistic objections; it merely postpones them. Specifically, when going beyond sentential logic and introducing quantification theory, I need that quantification of sentential functions produces sentences, i.e., if $\varphi(x)$ is a sentence for each value of x, then $\forall x\, \varphi(x)$ and $\exists x\, \varphi(x)$ are also sentences. According to intuitionists, this idea, that quantification produces sentences, may be reasonable for some informal notions of sentence, but it is entirely unjustified for my notion of sentence that requires decidability. Specifically, decidability of $\varphi(x)$ for each individual x is unlikely to imply decidability of $\forall x\, \varphi(x)$ or of $\exists x\, \varphi(x)$ unless the domain over which x ranges is finite. An infinite collection of x's cannot be surveyed, to check the truth value of each $\varphi(x)$ individually and thus determine the truth values of the quantified formulas.

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The purpose of this paper is to look more closely at the question of surveyability, in this sense, of sets. The framework of the investigation will be what is usually called intuitionistic type theory,[1] the sort of type theory that arises as the internal logic of elementary topoi [1,4,5]. In this framework, I propose to define two notions of surveyability and to study them individually, to compare them with each other, and to compare them with some related notions. The reason for two notions of surveyability is that there are two quantifiers under consideration; hence we have the following definitions, formulated in intuitionistic type theory. Ω is, as usual, the set of truth values, also called the subobject classifier and the power set of a singleton.

Definition 1. A set X is *universally surveyable* or $\forall$-*surveyable* if, whenever $\varphi : X \to \Omega$ satisfies $(\forall x \in X)(\varphi(x) \vee \neg\varphi(x))$, then it also satisfies $((\forall x \in X)\varphi(x)) \vee \neg((\forall x \in X)\varphi(x))$.

Definition 2. A set X is *existentially surveyable* or $\exists$-*surveyable* if, whenever $\varphi : X \to \Omega$ satisfies $(\forall x \in X)(\varphi(x) \vee \neg\varphi(x))$, then it also satisfies $((\exists x \in X)\varphi(x)) \vee \neg((\exists x \in X)\varphi(x))$.

Convention 3. Like the preceding two definitions, all the definitions, theorems, and corollaries in this paper are to be understood as assertions in intuitionistic type theory. In particular, "set" is to be understood as meaning a term of power set type $\mathcal{P}(X)$ for some type X. Thus, a theorem about "all sets" is formally a theorem schema, with one instance for each type. Similarly, "function" is to be understood as meaning a term of (possibly partial) function type.

The proofs of theorems and corollaries in this paper are informal arguments intended to be easily formalizable in intuitionistic type theory. The same goes for most of the other comments and arguments here (but see below concerning metatheory).

Thus, these arguments are valid in the internal logic of a topos, and the reader is welcome to adopt a topos-theoretic interpretation of our results. Note, however, that, if we refer to "all elements of a set X", this is to be understood as a universal quantifier in the internal logic of the topos (intuitionistic type theory), not as referring only to global elements, morphisms $1 \to X$ in the topos. Similarly, a statement about "all functions $X \to Y$" is to be understood in the internal logic, not as referring only to morphisms in the topos.

Although, as stated above, our theorems, corollaries, and definitions are formulated in intuitionistic type theory, some metatheoretical examples are to be understood in the context of classical mathematics. Here "metatheoretical" means referring to models for intuitionistic type theory, rather than working within that theory. The models in question are topoi, or Kripke models, or topological models, or Heyting algebras. Specifically, the last part of Example 7; Examples 8, 9, 10, and 11; and the last part of Remark 22 present, from a classical point of view, information about models of intuitionistic type theory, delimiting what is provable in that theory.

Remark 4. It is useful to observe that, in the definitions of both sorts of surveyability, we may assume that φ is defined not only on the set X under discussion but on some given superset of X. That is, if $X \subseteq Y$, then X is $\forall$-surveyable if and only if, whenever $\varphi : Y \to \Omega$ satisfies $(\forall x \in X)(\varphi(x) \vee \neg\varphi(x))$, then it also satisfies $((\forall x \in X)\varphi(x)) \vee \neg((\forall x \in X)\varphi(x))$. The analogous observation applies to $\exists$-surveyability. The reason, in both cases, is simply that Ω (or, more generally, any power set) is injective in the category-theoretic sense. That is, if $X \subseteq Y$ then any function $X \to \Omega$ is the restriction to X of some $Y \to \Omega$. Note, however, that these

[1] Intuitionists would probably prefer not to have the name of their philosophy attached to this theory, since the theory allows unrestricted formation of power sets; nevertheless, the name has become attached, in common usage, to this sort of type theory.