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Intuitionistic sequential compactness?

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Abstract

Perhaps because the classical notion of sequential compactness fails to apply constructively even to $\{0, 1\}$, Brouwer and his successors have paid little attention to the possibility of a constructive counterpart that is classically equivalent to sequential compactness and has serious potential for applications in analysis. We discuss such a notion – the anti-Specker property – and its equivalence, over Bishop-style constructive mathematics, to Brouwer's fan theorem for c-bars.

1. Introduction

In his breakthrough 1924 paper *Proof that every fully-defined function is uniformly continuous* [21], Brouwer introduced the notion of bar induction, leading to a proof of what, after WWII, became known as his fan theorem. This was then applied to proving that every – *every* – real-valued function defined throughout the closed interval [0, 1] is uniformly continuous, a result that (almost) in van Dalen's words 'proudly set intuitionism [apart] from classical mathematics'. Two years later, Brouwer used his fan theorem to bring within intuitionistic mathematics the Heine–Borel theorem: every open cover of [0, 1] contains a finite subcover [22]. This, in turn, brought the open-cover compactness property of metric spaces into the intuitionistic fold.

Classically, open-cover compactness holds for a metric space X if and only if the space is sequentially compact: every sequence in X has a convergent subsequence. This is not the case constructively (that is, in any mathematical framework in which, as in intuitionism, an object exists if and only if we can construct it): the sequential compactness of the open-cover-compact pair set $\{0, 1\}$ is equivalent to the **limited principle of omniscience**,

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LPO: *For every binary sequence $(a_n)_{n\geqslant 1}$, either $a_n = 0$ for all n or else there exists (we can construct/compute) n such that $a_n = 1$,*

which, for very good reasons (see pages 4–5 of [8]), is regarded as essentially nonconstructive.

It seems that neither Brouwer nor his followers – including constructivists, like Bishop, who do not adopt Brouwer's special intuitionistic principles like the fan theorem – produced an intuitionistically acceptable and useful sequential compactness property that is classically equivalent to sequential compactness. In this paper we discuss a constructive sequential compactness property that does have serious applications and that is related to a version of Brouwer's fan theorem. Before doing so, though, we need to clarify the constructive framework in which the paper lies, and provide some definitions for later use.

We work throughout in **(Bishop-style) constructive mathematics** (**BISH**): mathematics carried out with intuitionistic logic,[1] in an appropriate set- or type-theoretical foundational system such as can be found in [1,2,41], and including the principle of dependent choice. It is, however, important to appreciate from the outset that we work in the relatively informal style of the analyst, rather than within a formal system; formalisation would render the paper both excessively long and probably unappealing to the working constructive analyst.

Informally, we can think of

- intuitionistic mathematics (**INT**) as **BISH** plus Brouwer's continuity principle(s) and fan theorem;
- recursive constructive mathematics (**RUSS**) as **BISH** plus the Church–Markov–Turing thesis and Markov's principle of unbounded search; and
- classical mathematics (**CLASS**) as **BISH** plus the law of excluded middle and the full axiom of choice.

Thus **BISH** can be viewed as the common constructive core of **INT**, **RUSS,** and **CLASS**.

Every theorem of **BISH** is also one of **INT**, **RUSS**, and **CLASS**. In particular, if (like the Bolzano–Weierstraß theorem **BW**$'$ discussed in Section 2), a statement T holds classically but is false in either **INT** or **RUSS**, then T can be neither proved nor disproved within **BISH**. For more on the relation of **BISH** to **INT**, **RUSS**, and **CLASS**, see [4] or [19].

When we have a statement like

Theorem Every x is a y

it should be taken for granted that the theorem is proved within **BISH** unadorned by additional assumptions. But when we have a statement like

Theorem **ROT** $\vdash$ Every x is a y

we mean that 'Every x is a y' holds in **BISH** plus the additional principle **ROT** (whatever that may be); that is, within **BISH + ROT**.

2. Brouwer and Bolzano–Weierstrass

Perhaps the closest that Brouwer and others came to a viable constructive equivalent of sequential compactness is found in his investigation into the Bolzano–Weierstraß theorem,[2]

[1] Under the Brouwer–Heyting–Kolmogorov interpretation of the connectives and quantifiers [20,45].

[2] We denote by **R**, **N**, and $\mathbf{N}^+$ the sets of real numbers, natural numbers, and positive integers respectively.

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