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Univalence as a principle of logic

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Abstract

It is sometimes convenient or useful in mathematics to treat isomorphic structures as the same. The recently proposed Univalence Axiom for the foundations of mathematics elevates this idea to a foundational principle in the setting of homotopy type theory. It provides a simple and precise way in which isomorphic structures can be identified. We explore the motivations and consequences, both mathematical and philosophical, of making such a new logical postulate.

There are more things in heaven and earth, Horatio, than are dreamt of in your philosophy.
Hamlet (1.5.167-8)

0. Introduction

The Univalence Axiom has recently been proposed by the distinguished homotopy theorist Vladimir Voevodsky as a new principle for the foundations of mathematics. This axiom is formulated within a recently developed system of foundations called homotopy type theory, and, roughly speaking, permits isomorphic structures to be identified. The resulting picture of the mathematical universe is rather different from that corresponding to conventional set theoretic foundations. It is not entirely alien, however, and it even has many interesting connections to some traditional philosophical issues.

There are three main themes in what follows:

1. Recent advances in computer technology have given rise to the possibility of a new, computer-assisted, formalization of mathematics, breathing new life into the old Logicist program of Frege and Russell.

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2. One of the main tools for such formalizations is Martin-Löf type theory, which has recently been connected to homotopy theory, of all things. But why should these remote fields be related at all? As it turns out, there are good reasons why logic and homotopy are closely connected, and they have to do with some of the oldest problems of logic and analytic philosophy.
3. In homotopy type theory, the Univalence Axiom implies that isomorphic structures can be identified, in a certain, precise sense that I will explain. This sounds like the sort of thing that one hears from structuralists in the philosophy of mathematics, and indeed, the development of mathematics within this new system does have a distinctly structural character. But the Univalence Axiom is not just a practical device for implementing structuralism within a formal system of foundations: it has a purely logical justification as well. We consider this justification as support for taking Univalence as a new foundational *axiom*.

1. Why new foundations for mathematics?

There is at present a serious mismatch between the everyday practice of mathematics and the "official" foundations of mathematics represented by ZFC. This is evident in the fact that working mathematicians do not really pay attention to foundations, and even violate their restrictions whenever convenient. An example is the common practice of identifying isomorphic objects with devices like "abuse of notation", which is a harmless move, but one that is outside the scope of what can actually be represented within the foundational system. The fact that this is a sort of joke, and is often accompanied by a shrug, is just evidence that the mathematician is aware of the mismatch, but does not really care.

This particular tension between official foundations and everyday practice really goes back to Bourbaki's "structuralist" approach, according to which the real objects of mathematics are not specific number systems but "structures", like groups, modules, spaces, etc., which can be described by operations and relations in a hierarchy of sets. The official formulation, however, was formulated in simple type theory (not axiomatic set theory) in order to be able to give an associated notion of "isomorphism" of structures, which was needed in order to determine which statements and properties are really relevant — namely, those that respect the isomorphisms. Thus topology is the study of spaces "up to homeomorphism", group theory of groups "up to group isomorphism", etc. — as opposed to actual sets equipped with a system of open sets, a multiplication, and so on. In particular, e.g., the real numbers are not regarded as a specific set of sets (Dedekind cuts, say), but rather an "abstract" object (a field with certain properties), given by a particular presentation, but invariant under all isomorphisms between other possible presentations. This general methodology and point of view has now become quite pervasive in modern mathematics

Philosophers of mathematics have certainly noticed this trend, and they have developed various forms of "structuralism" based on the tools commonly available to them: second-order logic, modal logic, etc. But they have been hampered in understanding structuralism by the absence of a comprehensive system of foundations that fully incorporates a modern, structuralist viewpoint, in the way that set theory does for non-structuralist foundations. Some use of category theory has been made, to be sure, and this is certainly a step in the right direction. But it is fair to say that (until recently) no satisfactory, comprehensive, "structuralist" system of foundations had been formulated, either by philosophers, or by mathematicians themselves.

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