INDAG: 591 Model 1 pp. 1–9 (col. fig: NIL)

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ScienceDirect

Indagationes Mathematicae xx (xxxx) xxx–xxx

indagationes mathematicae

www.elsevier.com/locate/indag

Virtual Special Issue - L.E.J. Brouwer after 50 years

Real numbers and projective spaces: Intuitionistic reasoning with undecidable basic relations

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Abstract

Brouwer introduced in 1924 the notion of an apartness relation for real numbers, with the idea that whenever it holds, a finite computation verifies it in contrast to equality. The idea was followed in Heyting's axiomatization of intuitionistic projective geometry. Brouwer in turn worked out an intuitionistic theory of "virtual order." It is shown that Brouwer's proof of the equivalence of virtual and maximal order goes only in one direction, and that Heyting's axiomatization needs to be made a bit stronger.

1. Decidability: a common misconception about intuitionism

In Emile Borel's version of constructivism from the first decade of the 20th century, there is a rather clear recognition that the equality of real numbers cannot be a decidable relation: For example, there is a way of computing what is known as Riemann's constant C, and the computation has so far given $0.4999\ldots$. If every successive decimal is 9, we have $C = 0.5$, otherwise $\neg\, C = 0.5$ holds. The condition for equality is expressed as a universal quantification over the decimals.

The same insight as in Borel got a more forceful expression in L. Brouwer's ideas about real numbers in the 1920s: He replaced equality of real numbers as a basic notion with the *apartness* of two reals, written $a \neq b$. That a and b are in this way distinct requires that there is a positive lower bound for their difference. Thus, a finite determination of values will verify apartness, though not falsify it, the precise contrary to the case of equality. The latter notion can now be defined as the negation of apartness. The point is how to reason with ideal objects and concepts such as real numbers and their properties and relations. If we follow the ideas of

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https://doi.org/10.1016/j.indag.2017.10.012

the logical positivists fashionable in the 1920s, there should be no undecidable basic relations: Their doctrine of verificationism requires a method for deciding truth, otherwise a notion is not meaningful. If such a method is assumed for the equality of reals, the law of excluded middle is justified: Then, its use gives $C = 0.5 \vee \neg C = 0.5$ with C Riemann's constant and there are two cases, with two different consequences. As long as the value of C remains undecided, nothing concrete follows from the cases. If instead the classical law is not allowed to enter, computability is maintained: logical reasoning will never lead from assumptions with a finitary meaning into something infinitistic. That is the whole point of constructive or intuitionistic logic.

The standard view in the 1930s was that finitism and constructivism contain the requirement by which all basic relations be decidable and all functions computable. The former is an erroneous view to which one however could easily be led if one considered only intuitionistic arithmetic that can be formulated with a decidable equality as the only primitive relation, instead of also the intuitionistic theory of real numbers that cannot be based on a decidable equality, or the first intuitionistic axiomatization ever, Arend Heyting's [4] system of intuitionistic projective geometry in which decidability of the basic apartness relations cannot be assumed. One of the first outside the intuitionist camp to realize the difference was Kurt Gödel whose lectures on intuitionism in Princeton in 1941 have been preserved in manuscript form. The extant text begins with two pages of improvements for the lectures, written in his Gabelsberger shorthand. One of the improvements is: "The belief is put aside that a system of axioms has an intuitionistic sense only if the basic concepts are decidable".

From the actual formal work of the intuitionists, Brouwer and his student Heyting in the first place, it can be seen that they required all functions to be computable, but not all basic relations to be decidable. So, why was there such a belief or requirement? One reason lies in the possibility to emulate operations with added basic relations. For example, one could substitute the operation of sum in arithmetic by a three-place relation written, say, $\Sigma(a, b, c)$ with the intended meaning that c is the sum of a and b.

2. Apartness relations

Brouwer's [1] *Intuitionistic division of the basic notions of mathematics* contains a replacement of the equality relation of two real numbers by *apartness*, $a \neq b$. The properties of apartness are, in logical notation:

1. $\neg\, a \neq a$ *irreflexivity*
2. $a \neq b \supset a \neq c \vee b \neq c$ *apartness axiom, co-transitivity*

The second axiom is notable in a constructive context because it has a disjunction in a part of the formula (positive part) that cannot be rewritten in a constructively equivalent way without disjunction. It follows that whenever we have established $a \neq b$, any third real number c can be taken and the two cases $a \neq c$ and $b \neq c$ formed.

By putting a for c in the second axiom, we get $a \neq b \supset a \neq a \vee b \neq a$, with the first disjunct negated in axiom 1. Therefore symmetry, $a \neq b \supset b \neq a$, follows.

Equality is a defined notion:

$$a = b \equiv \neg\, a \neq b$$

Reflexivity of equality is immediate from the definition, and symmetry and transitivity follow as contrapositions of symmetry of apartness and of axiom 2, the latter in the "Euclidean" form $a = c \,\&\, b = c \supset a = b$, by Euclid's axiom in the *Elements* that says: "Two things equal to a third are equal among themselves".