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A cubical model of homotopy type theory ☆

Steve Awodey

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ABSTRACT

We construct an algebraic weak factorization system (L, R) on the category of cartesian cubical sets, in which the canonical path object factorization $A \to A^{\mathrm{I}} \to A \times A$, induced by the 1-cube I, is an (L, R)-factorization for any R-object A.

Introduction

The main goal of this note is to prove the following:

Theorem. *There is an algebraic weak factorization system* (L, R) *on the category of cartesian cubical sets such that for any* R*-object* A*, the factorization of the diagonal map,*

$$A \to A^{\mathrm{I}} \to A \times A\,,$$

determined by the 1-cube I*, is an* (L, R)*-factorization.*

The motivation for this result is that it follows that there is a cubical model of basic homotopy type theory in which the identity type of a type A can be taken to be the path-object A^{I}. We shall merely sketch this application to type theory, leaving the (somewhat delicate) details of the actual model construction to other sources (such as [14,12]). The question of whether there can be such a cubical model, in which the

☆ This paper provides the details of a result that was mentioned in my 2015 LC/CLMPS lecture in Helsinki. The presentation derives from a series of lectures given in May–June, 2016, in the Stockholm Logic Seminar and the Topological Activities Seminar. Thanks are due to the organizers of the Helsinki meeting, and to Erik Palmgren and the members of the Stockholm Logic Group.

E-mail address: awodey@cmu.edu.

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identity type is interpreted as a path space of the form A^{I}, arises because in some recent work (including [4,6]) the so-called computation rule for identity types fails under this interpretation.

We begin by reviewing the basic idea of homotopical semantics of type theory in weak factorization systems, including the somewhat technical issue of coherence that motivates the use of algebraic weak factorization systems. We then explain the use of cubical sets and construct the desired algebraic weak factorization system.

Some of our development is motivated by the recent work [4], and indeed part of the purpose of this note is to analyze certain aspects of that work. In other respects, however, our approach is independent and distinct.

1. The basic homotopical interpretation

Definition 1. A *weak factorization system* on a category $\mathbb{C}$ consists of two classes of arrows,

$$\mathcal{L} \hookrightarrow \mathbb{C}_1 \hookleftarrow \mathcal{R}$$

satisfying the following conditions:

1. Every arrow $f : X \to Y$ in $\mathbb{C}$ factors as a left map followed by a right map,

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\mathcal{L}"'] & & Y \\ & \cdot \arrow[ur, "\mathcal{R}"'] & \end{tikzcd}$$

2. Given any commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\mathcal{L}"'] & B \arrow[d, "\mathcal{R}"] \\ C \arrow[r] \arrow[ur, dotted] & D \end{tikzcd}$$

with an $\mathcal{L}$-map on the left and an $\mathcal{R}$-map on the right, there is a "diagonal filler" as indicated, making both triangles commute.
3. Each of the classes $\mathcal{L}$ and $\mathcal{R}$ is closed under retracts in the arrow category $\mathbb{C}^{\to}$.

Examples include (i) Groupoids (or categories), with $\mathcal{R}$ = isofibrations and $\mathcal{L}$ = injective equivalences; (ii) Simplicial sets, with $\mathcal{R}$ = Kan fibrations and $\mathcal{L}$ = acyclic cofibrations. Since a Quillen model structure on a category by definition involves two interrelated such weak factorization systems, this provides many examples as well as the basic homotopical intuition. In a WFS, we may think of the $\mathcal{R}$-maps as "fibrations", i.e. good families of objects indexed by the codomain. The basic idea of the homotopy interpretation is to use these as the dependent types.

1.1. Interpreting Id*-types*

We want to interpret (intensional) Martin-Löf type theory, as specified e.g. in [13]; however, our focus here is entirely on the interpretation of identity types, and even then, we are merely giving a sketch of the intended approach. For detailed discussions of related interpretations of the full system, with Σ and Π-types, see e.g. [14,12]. The original source of the homotopical interpretation of identity types is [2].