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A canonical hidden-variable space [☆]

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A B S T R A C T

The hidden-variable question is whether or not various properties — randomness or correlation, for example — that are observed in the outcomes of an experiment can be explained via introduction of extra (hidden) variables which are unobserved by the experimenter. The question can be asked in both the classical and quantum domains. In the latter, it is fundamental to the interpretation of the quantum formalism (Bell [1], Kochen and Specker [9], and others). In building a suitable mathematical model of an experiment, the physical set-up will guide us on how to model the observable variables — i.e., the measurement and outcome spaces. But, by definition, we cannot know what structure to put on the hidden-variable space. Nevertheless, we show that, under a measure-theoretic condition, the hidden-variable question can be put into a canonical form. The condition is that the σ-algebras on the measurement and outcome spaces are countably generated. An argument using a classical result on isomorphisms of measure algebras then shows that the hidden-variable space can always be taken to be the unit interval equipped with the Lebesgue measure on the Borel sets.

1. Introduction

Consider an experiment in which Alice can make one of several measurements on her part of a certain system and Bob can make one of several measurements on his part of the system. Each pair of measurements (one by Alice and one by Bob) leads to a pair of outcomes (one for Alice and one for Bob). We keep track of the frequency distribution of the different pairs of outcomes that arise. This situation can be abstracted

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to an **empirical model**, which, for each pair of measurements, specifies a probability measure on pairs of outcomes.

An associated **hidden-variable model** is obtained by starting with the empirical model and then appending to it extra variables that are assumed to be present in a more complete theory of how the data are generated. The uses of hidden-variable analysis include: (i) seeing if a deterministic account can be given of the observed data, and (ii) seeing if a common-cause account can be given of correlations in the observed data.

Arguably, the most famous context for hidden-variable analysis is quantum mechanics (QM). Starting with von Neumann [13], and including, most famously, Einstein, Podolsky, and Rosen [6], Bell [1], and Kochen and Specker [9], a vast literature has grown up around the question of whether or not a hidden-variable formulation of QM is possible. The watershed no-go theorems of Bell and Kochen–Specker give conditions under which the answer is no.

Hidden variables are variables above and beyond those which are part of the actual experiment, and are therefore unobserved. This poses a question: *What can one assume about the structure of the space on which a hidden variable lives?* Choosing a good empirical model includes choosing measurement and outcome spaces that incorporate appropriate physical features (say, discreteness, connectedness, or other features). But, since it is unobserved, there is no such guide to choosing a hidden-variable space.

This may or may not be a serious obstacle. The question often under study is whether or not a hidden-variable model exists that exhibits desired properties such as determinism or common-cause correlation. For a positive answer, we may be satisfied with showing that, at least for a certain choice of hidden-variable space, such a model exists. But, some of the most important results — including the Bell and Kochen–Specker theorems — are negative answers, asserting that no hidden-variable model with certain properties exists. For a non-existence result to be definitive, we need to search over all (not just some) hidden-variable spaces.

Our main result is that there is a **canonical hidden-variable model**. To be more specific, fix an empirical model. Suppose there is an associated hidden-variable model that yields, for each pair of measurements, the same probability measure over joint outcomes. We will say that the hidden-variable model **realizes** the empirical model. We want to know if we can put this hidden-variable model into a canonical form. More than this, to be useful, such a canonical hidden-variable model must preserve properties — determinism, common-cause correlation, etc. — satisfied by the original hidden-variable model.

We show that, under a measure-theoretic condition, such a canonical model exists: *If the σ-algebras on the measurement and outcome spaces are countably generated, then the hidden-variable space can always be taken to be the unit interval equipped with the Lebesgue measure on the Borel sets.* Note that if a probability space has a countably generated σ-algebra, then the associated probability algebra is separable. The key to our result is the classical theorem that any two separable atomless probability algebras are isomorphic. This theorem can be found in Carathéodory [4] and Halmos and von Neumann [7]. It is also a special case of Maharam's Theorem [10].

2. Empirical and hidden-variable models

Alice has a space of possible measurements, which is a measurable space $(Y_a, \mathcal{Y}_a)$, and a space of possible outcomes, which is a measurable space $(X_a, \mathcal{X}_a)$. Likewise, Bob has a space of possible measurements, which is a measurable space $(Y_b, \mathcal{Y}_b)$, and a space of possible outcomes, which is a measurable space $(X_b, \mathcal{X}_b)$. We will restrict attention to bipartite systems. (We comment later on the extension to more than two parts.) There is also a hidden-variable space, which is an unspecified measurable space $(\Lambda, \mathcal{L})$. Write

$$(X, \mathcal{X}) = (X_a, \mathcal{X}_a) \otimes (X_b, \mathcal{X}_b),$$

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