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Atomic polymorphism and the existence property

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ABSTRACT

We present a purely proof-theoretic proof of the existence property for the full intuitionistic first-order predicate calculus, via natural deduction, in which commuting conversions are not needed. Such proof illustrates the potential of an atomic polymorphic system with only three generators of formulas – conditional and first and second-order universal quantifiers – as a tool for proof-theoretical studies.

1. Introduction

In [5] it was shown that there is a purely proof-theoretic proof of the disjunction property for the full intuitionistic propositional calculus (**IPC**), via natural deduction without the need of commuting conversions. Such a proof relies in a sound and faithful embedding of **IPC** into an atomic polymorphic system with only two generators of formulas – conditional and second-order universal quantifier – denoted by $\mathbf{F_{at}}$.

Is it possible to give an alternative proof of the existence property for the full intuitionistic first-order predicate calculus (**IQC**) in the above *commuting conversion-free* manner? We show in this paper that the answer is yes. The atomic polymorphic system used to interpret **IQC**, which we denote by $\mathbf{F_{at}^{Q}}$, was

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introduced in [3] under the designation of atomic QSOL^i. Such system, which we review in the section below, has three generators of formulas, the conditional and the first and second-order universal quantifiers.

Using atomic polymorphic systems to avoid complications of the natural deduction calculus is not a novelty. The defects of the latter calculus were eloquently exposed in [8]. According to Jean-Yves Girard *et al.* (page 74) 'The elimination rules [for $\bot$, $\vee$ and $\exists$] are very bad.' and proceed 'Indeed, we cannot decently work with the full fragment without identifying *a priori* different deduction... the need to add these supplementary rules [referring to the commuting conversions for $\bot$, $\vee$ and $\exists$] reveals an inadequacy of the syntax.' Fernando Ferreira coined a simple and elegant way of avoiding the 'inadequacy' of the natural deduction calculus in the context of **IPC** presenting in [2] an embedding of **IPC** into $\mathbf{F_{at}}$[2] – a predicative system without *bad* connectives. Several studies in **IPC** taking advantage of its embedding into $\mathbf{F_{at}}$ followed. See for example [4–7]. F. Ferreira and the author showed in [3] that the previous embedding could be extended to the full intuitionistic first-order predicate calculus. More precisely, it is possible to embed **IQC** into an atomic polymorphic calculus ($\mathbf{F^Q_{at}}$) with implication and first and second-order universal quantifiers, i.e., still without *bad* connectives. In the present paper, having in view to show that the above embedding into $\mathbf{F^Q_{at}}$ can be a useful tool for **IQC**, we prove the faithfulness of such embedding and explore some proof-theoretic properties of $\mathbf{F^Q_{at}}$. Namely, we show that if the translation of $\exists x.A$ is derivable in $\mathbf{F^Q_{at}}$ then there is a first-order term t such that the translation of $A[t/x]$ is derivable in $\mathbf{F^Q_{at}}$. Note that in order to obtain t we do not rely on commuting conversions. As an illustration of the embedding's potential we show that the existence property for **IQC** can be easily derived from the homologous property in the atomic polymorphic context. Not surprisingly, in this alternative proof of the existence property for **IQC** commuting conversions are absent.

The paper is organized as follows: in Section 2 we review the atomic polymorphic system $\mathbf{F^Q_{at}}$ and the sound translation of **IQC** into $\mathbf{F^Q_{at}}$ and we prove some properties used in following sections such as strong normalization and faithfulness for $\mathbf{F^Q_{at}}$; in Section 3, we prove that $\mathbf{F^Q_{at}}$ enjoys the existence property and in Section 4 we present an alternative proof of the existence property for **IQC**, via natural deduction, without commuting conversions.

2. Atomic polymorphism for IQC

The atomic polymorphic calculus used in this paper is $\mathbf{F^Q_{at}}$, a calculus described in [3],[3] which has first and second-order constants, first-order variables x, y, z, ..., sentential variables X, Y, Z, ... and the connectives: $\to$ (conditional), $\forall_1$ (first-order universal quantifier) and $\forall_2$ (second-order universal quantifier). *Atomic formulas* are either second-order constants and variables or expressions of the form $P(t_1, \ldots, t_n)$, where P is a n-ary relational symbol and $t_1, \ldots, t_n$ are first-order terms. The class of *formulas* is built in the usual way from atomic formulas, closing the class under $\to$, $\forall_1$ and $\forall_2$. I.e. if A and B are formulas, x is a first-order variable and X is a second-order variable then $A \to B$, $\forall_1 x.A$ and $\forall_2 X.A$ are also formulas. The subscripts in $\forall_1$ and $\forall_2$ are usually omitted. In what follows, the language of $\mathbf{F^Q_{at}}$ will be denoted by $\mathcal{L}$.

The logic of $\mathbf{F^Q_{at}}$ is *intuitionistic* logic that we formulate in the natural deduction calculus through the usual introduction and elimination rules for $\to$, $\forall_1$ and $\forall_2$:

$$\dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \to B}\ {\to}\mathrm{I} \qquad \dfrac{\begin{matrix}\vdots \\ A\end{matrix}}{\forall x.A}\ \forall_1\mathrm{I} \qquad \dfrac{\begin{matrix}\vdots \\ A\end{matrix}}{\forall X.A}\ \forall_2\mathrm{I}$$

where x and X do not occur free in any undischarged hypothesis (respectively),

[2] In paper [2] $\mathbf{F_{at}}$ was designated by atomic PSOL^i.
[3] In [3], system $\mathbf{F^Q_{at}}$ was denoted by atomic QSOL^i.

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