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Virtual large cardinals

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A B S T R A C T

We introduce the concept of virtual large cardinals and apply it to obtain a hierarchy of new large cardinal notions between ineffable cardinals and $0^{\#}$. Given a large cardinal notion $\mathcal{A}$ characterized by the existence of elementary embeddings $j : V_\alpha \to V_\beta$ satisfying some list of properties, we say that a cardinal is *virtually* $\mathcal{A}$ if the embeddings $j : V_\alpha^V \to V_\beta^V$ exist in the generic multiverse of V. Unlike their ideological cousins generic large cardinals, virtual large cardinals are actual large cardinals that are compatible with $V = L$. We study virtual versions of extendible, n-huge, and rank-into-rank cardinals and determine where they fit into the large cardinal hierarchy.

1. Introduction

The current paper introduces the theory of virtual large cardinals. Suppose $\mathcal{A}$ is a large cardinal notion that can be characterized by the existence of elementary embeddings $j : V_\alpha \to V_\beta$ satisfying some list of properties. For instance, both extendible cardinals and I3 cardinals meet these requirements. Recall that κ is *extendible* if for every $\alpha > \kappa$, there is an elementary embedding $j : V_\alpha \to V_\beta$ with critical point κ and $j(\kappa) > \alpha$, and recall also that κ is I3 if there is an elementary embedding $j : V_\lambda \to V_\lambda$ with critical point $\kappa < \lambda$. Let us say that a cardinal κ is *virtually* $\mathcal{A}$ if the embeddings $j : V_\alpha \to V_\beta$ needed to witness $\mathcal{A}$ can be found in set-generic extensions of the universe V; equivalently we can say that the embeddings exist in the generic multiverse of V. Indeed, as we shall see in Section 3, it suffices to only consider the collapse extensions. So we now have that κ is *virtually extendible* if for every $\alpha > \kappa$, some set-forcing extension has an elementary embedding $j : V_\alpha^V \to V_\beta^V$ with critical point κ and $j(\kappa) > \alpha$, and we have that κ is *virtually*

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I3 if some set-forcing extension has an elementary embedding $j : V_\lambda^V \to V_\lambda^V$ with critical point κ. As we will see in Section 2 the template of virtual large cardinals can be applied to several large cardinals notions in the neighborhood of a supercompact cardinal. We can even apply it to inconsistent large cardinal principles to obtain virtual large cardinals that are compatible with $V = L$.

The concept of virtual large cardinals is close in spirit to generic large cardinals, but is technically very different. Suppose $\mathcal{A}$ is a large cardinal notion characterized by the existence of elementary embeddings $j : V \to M$ satisfying some list of properties. Then we say that a cardinal κ is *generically* $\mathcal{A}$ if the embeddings needed to witness $\mathcal{A}$ exist in set-forcing extensions of V. More precisely, if the existence of $j : V \to M$ satisfying some properties witnesses $\mathcal{A}$, then we want a forcing extension $V[G]$ to have a definable $j : V \to M$ with these properties, where M is an inner model of $V[G]$. So for example, κ is *generically supercompact* if for every $\lambda > \kappa$, some set-forcing extension $V[G]$ has an elementary embedding $j : V \to M$ with critical point κ and $j''\lambda \in M$. If κ is not actually λ-supercompact, the model M will not be contained in V. Most generic large cardinals are either known to have the same consistency strength as their actual counterparts or are conjectured to have the same consistency strength based on currently available evidence.[1] Most importantly, generic large cardinals need not be actually "large" since, for instance, ω_1 can be generically supercompact.

In the case of virtual large cardinals, because we consider only set-sized embeddings, the source and target of the embedding are both from V, and because the embedding exists in a forcing extension, there is no a priori reason why the target model would have any closure at all. The combination of these gives that virtual large cardinals are actual large cardinals that fit into the large cardinal hierarchy between ineffable cardinals and $0^\#$. If $0^\#$ exists, the Silver indiscernibles have all the virtual large cardinal properties we consider in this article, and all these notions will be downward absolute to L.

The first virtual large cardinal notion, the remarkable cardinal, was introduced by the second author in [15]. A cardinal κ is *remarkable* if for every $\lambda > \kappa$, there is $\bar\lambda < \kappa$ such that in a set-forcing extension there is an elementary embedding $j : V_{\bar\lambda}^V \to V_\lambda^V$ with $j(\mathrm{crit}(j)) = \kappa$. It turns out that remarkable cardinals are virtually supercompact because, as shown by Magidor [13], κ is supercompact precisely when for every $\lambda > \kappa$, there is $\bar\lambda < \kappa$ and an elementary embedding $j : V_{\bar\lambda} \to V_\lambda$ with $j(\mathrm{crit}(j)) = \kappa$. The second author showed that the existence of a remarkable cardinal is equiconsistent with the assertion that the theory of $L(\mathbb{R})$ cannot be changed by proper forcing [15], and since then it has turned out that remarkable cardinals are equiconsistent to other natural assertions such as the third-order Harrington's principle [3].

The idea behind the concept of virtual large cardinals of taking a property characterized by the existence of elementary embeddings of sets and defining a virtual version of the property by positing that the embeddings exist in the generic multiverse can be extended beyond large cardinals. In [2], together with Bagaria, we studied a virtual version of Vopěnka's Principle (Generic Vopěnka's Principle) and a virtual version of the Proper Forcing Axiom PFA. Fuchs has generalized this approach to obtain virtual versions of other forcing axioms such as the forcing axiom for subcomplete forcing SCFA [8] and resurrection axioms [9]. Each of these virtual properties has turned out to be equiconsistent with some virtual large cardinal, which has so far been the main application of these ideas.

In Section 2, we will formally define several virtual large cardinal notions that we are going to study in this article. In Section 3, we will recall some standard absoluteness results about countable structures, using which we will, in particular, get useful reformulations of the definitions of virtual large cardinals. In Section 4, we will show where the virtual large cardinals we defined fit into the existing hierarchy. In Section 5, we will review some current applications of these ideas. Finally, in Section 6, we will briefly motivate the virtual large cardinal template we have chosen, by discussing some alternative definitions.

[1] Some care is necessary here, though. By a theorem of Shelah (cf. e.g. [16] for a writeup), a Woodin cardinal can be used to obtain that ω_1 is generically almost huge.

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