J. Math. Anal. Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Existence and uniqueness of solutions to singular Cahn–Hilliard equations with nonlinear viscosity terms and dynamic boundary conditions ☆

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A R T I C L E   I N F O

Article history:
Received 24 May 2018
Available online xxxx
Submitted by D.M. Ambrose

Keywords:
Cahn–Hilliard system
Dynamic boundary conditions
Nonlinear viscosity
Existence of solutions
Uniqueness

A B S T R A C T

We prove global existence and uniqueness of solutions to a Cahn–Hilliard system with nonlinear viscosity terms and nonlinear dynamic boundary conditions. The problem is highly nonlinear, characterized by four nonlinearities and two separate diffusive terms, all acting in the interior of the domain or on its boundary. Through a suitable approximation of the problem based on abstract theory of doubly nonlinear evolution equations, existence and uniqueness of solutions are proved using compactness and monotonicity arguments. The asymptotic behaviour of the solutions as the diffusion operator on the boundary vanishes is also shown.

1. Introduction

The viscous Cahn–Hilliard equation can be written in its general form as

$$\partial_t u - \Delta\mu = 0\,, \qquad \mu = \alpha(\partial_t u) - \Delta u + \beta(u) + \pi(u) - g \qquad \text{in } (0,T)\times\Omega\,,$$

where the unknown u and μ represent the so-called order parameter and chemical potential, respectively. Such equation is fundamental in the phase-separation of a binary alloy, for example, and describes important qualitative behaviour like the so-called spinodal decomposition: we refer to the classical works [3,25,26,30] for a physical derivation of the model and some studies on the spinodal decomposition process. Here, Ω is smooth bounded domain in $\mathbb{R}^N$ $(N = 2, 3)$ with smooth boundary Γ, and $T > 0$ is the final time. As usual, the term $\beta + \pi$ represents the derivative of a double-well potential, g is a given source and α is a monotone

☆ **Acknowledgments.** The author is very grateful to Pierluigi Colli for his expert support and fundamental advice. The author is also thankful for the warm hospitality and excellent working conditions at the Dipartimento di Matematica "F. Casorati", Università di Pavia (Italy), where a part of this work was written.

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https://doi.org/10.1016/j.jmaa.2018.09.034

function acting on $\partial_t u$. While in the original model α is a linear function, some generalizations have been proposed where the behaviour of α is of nonlinear type: see in this direction [22].

In the present contribution, we study the equation above coupled with the homogeneous Neumann boundary condition for μ

$$\partial_{\mathbf{n}}\mu = 0 \qquad \text{in } (0,T)\times\Gamma\,,$$

which is very natural and ensures the conservation of the mass in the bulk, and a second-order doubly nonlinear dynamic boundary condition for u

$$\alpha_\Gamma(\partial_t u) + \partial_{\mathbf{n}}u - \varepsilon\Delta_\Gamma u + \beta_\Gamma(u) + \pi_\Gamma(u) = g_\Gamma \qquad \text{in } (0,T)\times\Gamma\,.$$

Here, $\varepsilon > 0$ is a fixed constant, Δ_Γ is the usual Laplace–Beltrami operator on Γ, g_Γ is a prescribed source on the boundary and the term $\beta_\Gamma + \pi_\Gamma$ represents the derivative of a double-well potential on the boundary, which may possibly differ from the one in the interior of the domain Ω. Similarly, α_Γ is a generic monotone function. Dynamic boundary conditions have been recently proposed by physicists in order to take into account also possible interactions with the walls of a confined system: for a physical motivation of this choice and some studies on parabolic-type equations with dynamic boundary conditions we mention the works [15,24] and [16–18].

Cahn–Hilliard equations with dynamic boundary have been widely studied in the last years in the classical setting in which the viscosity terms depend linearly on the time-derivative of the order parameter. This framework corresponds in our notation to the choices $\alpha = aI$ and $\alpha_\Gamma = bI$, with $a, b > 0$ given constants and I the identity on $\mathbb{R}$. Let us mention in this direction the works [6–8,11,19,20,28] dealing with well-posedness, regularity, long-time behaviour of solutions and asymptotics, [5,9,10] for some corresponding optimal control problems, and [4,12] focused specifically on Allen–Cahn equations.

On the other hand, an important area of interest has been equally developed on the study of Cahn–Hilliard equations with possibly nonlinear viscosity terms: the reader can refer to the contributions [27] for existence-uniqueness and long-time behaviour under classical homogeneous Neumann conditions, and to [1] for a detailed thermodynamical derivation of the model and well-posedness in the case of Dirichlet conditions for the chemical potential. Let us also mention the work [29] dealing with a doubly nonlinear Cahn–Hilliard equation with a different type of nonlinearity in the viscosity, and the classical contributions [13,31] on a variational approach to abstract doubly nonlinear equations. As the reader may notice, in this case the attention is mainly focused on the presence of a double nonlinearity in the governing equation, and, consequently, the prescription on the boundary conditions remains quite broad and classical (homogeneous Neumann or Dirichlet type).

The aim of this paper is to provide some unifying existence and uniqueness results for the more general case when both dynamic boundary conditions and nonlinear viscosity terms are present in the system. From the physical point of view, the presence of dynamic boundary conditions and nonlinear viscosity terms is more accurate, and allows for a more genuine description on the process. On the other side, from the mathematical perspective, the model is much more difficult to handle and to study. Indeed, this specific description gives rise to a system with 4 nonlinearities: α and α_Γ acting on the time-derivatives and representing the viscosities, and β and β_Γ acting on the order parameter. Besides the non-triviality of the model, the presence of several nonlinearities is strongly stimulating and challenging. In order to include also possibly non-smooth potentials in our analysis, the nonlinearities are assumed to be possibly multivalued (maximal monotone) graphs.

To summarize, we are concerned with the following system

$$\partial_t u - \Delta\mu = 0 \qquad \text{in } (0,T)\times\Omega\,, \tag{1.1}$$

Please cite this article in press as: L. Scarpa, Existence and uniqueness of solutions to singular Cahn–Hilliard equations with nonlinear viscosity terms and dynamic boundary conditions, J. Math. Anal. Appl. (2018), https://doi.org/10.1016/j.jmaa.2018.09.034