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Three solutions for a nonlocal problem with critical growth

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ABSTRACT

The main goal of this work is to prove the existence of three different solutions (one positive, one negative and one with nonconstant sign) for the equation $(-\Delta_p)^s u = |u|^{p_s^*-2}u + \lambda f(x,u)$ in a bounded domain with Dirichlet condition, where $(-\Delta_p)^s$ is the well known p-fractional Laplacian and $p_s^* = \frac{np}{n-sp}$ is the critical Sobolev exponent for the non local case. The proof follows the ideas of [28] and is based in the extension of the Concentration Compactness Principle for the p-fractional Laplacian [20] and Ekeland's variational Principle [7].

1. Introduction

Let us consider the following non local equation with Dirichlet boundary conditions

$$\begin{cases} (-\Delta_p)^s u = |u|^{p_s^*-2}u + \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.1}$$

Where $s \in (0,1)$, Ω is a smooth and bounded domain in $\mathbb{R}^n$ and $(-\Delta_p)^s u$, called the p-fractional Laplacian, is defined up to a normalization constant by

$$(-\Delta_p)^s u := 2 \lim_{\varepsilon \to 0^+} \int\limits_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+ps}}\, dy.$$

When $p = 2$ this is the well known fractional Laplacian. Problems involving non local operators have many applications, just to cite a few, we refer to [6,8,13] for some physical models, [1,16,23] for some applications

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in finances, [3] for applications in fluid dynamics, [15,19,22] for application in ecology and [14] for some applications in image processing.

The functional framework for this operator are the fractional order Sobolev spaces, see [30] and [5]. The fractional order Sobolev space is defined by

$$W^{s,p}(\mathbb{R}^n) := \{u \in L^p(\mathbb{R}^n) \colon [u]_{s,p} < \infty\},$$

where $[u]_{s,p}$ is the famous seminorm of Gagliardo is defined by

$$[u]_{s,p} := \left(\int_{\mathbb{R}^{2n}} \frac{(u(x) - u(y))^p}{|x-y|^{n+ps}}\, dx\, dy \right)^{\frac{1}{p}},$$

and $W^{s,p}_0(\Omega)$ is defined by $W^{s,p}_0(\Omega) := \{u \in L^p(\mathbb{R}^n) : [u]_{s,p} < \infty, u = 0 \text{ in } \mathbb{R}^n \setminus \Omega\}$. It is well-known that when $sp < n$ the following Sobolev inequality holds

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{np}{n-sp}}\, dx \right)^{\frac{n-sp}{n}} \leq C \int_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}}\, dxdy$$

for $u \in C^\infty_c(\mathbb{R}^n)$, where $p^*_s = \frac{np}{n-sp}$ is called the critical Sobolev exponent. So, the embedding $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \leq q \leq p^*_s$ is continuous. Moreover, is compact for $1 \leq q < p^*_s$. Critical equations with the fractional Laplacian in bounded domains have been considered in [2,24–27]. Multiplicity of solutions for nonlocal equation with critical growth was studied in [11,21]. The main goal of this paper is to show the existence of three different solutions of the problem (1.1). Moreover these solutions are one positive, one negative and one with non constant sign. We impose adequate conditions on the source f and on the parameter λ but we do not impose any parity conditions on the source f. This result extends an old paper of Struwe [29]. Similar results for some local operators can be found in [4,28,9,17]. The method in the proof used in [29] consists on restricting the functional associated to (1.1) to three different manifolds constructed by imposing a sign restriction and normalizing condition. Then using Ekeland variational principle (see [7]) and a generalization to the fractional setting obtained by Mosconi et al. for any $1 < p < \frac{n}{s}$ (see [20]) of the well known Concentration Compactness Principle of P.L. Lions (see [18]), we can prove the existence of a critical point of each restricted functional, that are critical points of the unrestricted one.

Throughout this work, by weak solution of (1.1) we understand critical points of the associated energy functional acting on the Sobolev space $W^{s,p}_0(\Omega)$:

$$\Phi(u) = \frac{1}{p} \int_{\mathbb{R}^{2n}} \frac{(u(x) - u(y))^p}{|x-y|^{n+ps}}\, dy\, dx - \int_{\Omega} \frac{1}{p^*_s} |u(x)|^{p^*_s} + \lambda F(x, u(x))\, dx, \tag{1.2}$$

where $F(x,u) = \int_0^u f(x,z)dz$.

2. Assumptions and statement of the results

The precise assumptions on the source terms f are as follows:

(H1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$, is a measurable function with respect to the first argument and continuously differentiable with respect to the second argument for almost every $x \in \Omega$. Moreover, $f(x, 0) = 0$ for every $x \in \Omega$.

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