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On the set of shadowable measures

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ABSTRACT

We first prove that the set of shadowable measures of a homeomorphism of a compact metric space X is an $F_{\sigma\delta}$ set of the space $\mathcal{M}(X)$ of Borel probability measures of X equipped with the weak*-topology. Next that the set of shadowable measures is dense in $\mathcal{M}(X)$ if and only if the set of shadowable points is dense in X. Therefore, if X has no isolated points and every non-atomic Borel probability measure has the shadowing property, then the shadowable points are dense in X (this is false when the space has isolated points). Afterwards, we consider the *almost shadowable measures* (measures for which the shadowable point set has full measure) and prove that all of them are weak* approximated by shadowable ones. In addition the set of almost shadowable measures is a G_δ set of $\mathcal{M}(X)$. Furthermore, the closure of the shadowable points is the union of the supports of the almost shadowable measures. Finally, we prove that every almost shadowable measure can be weak* approximated by ones with support equals to the closure of the shadowable points.

1. Introduction

A well known result in measure theory is that the set of nonatomic Borel probability measures of a compact metric space X is a G_δ in $\mathcal{M}(X)$, the space of all Borel probability measures equipped with the weak* topology. It is known also that the set of ergodic invariant measures of a homeomorphism of X is also a G_δ set of $\mathcal{M}(X)$ ([10]). More recently, it was proved that the set of expansive measures of a homeomorphism of X is a $G_{\delta\sigma}$ set of $\mathcal{M}(X)$ ([7]). In light of these results it is natural to ask whether the same property holds for the set of shadowable measures of a homeomorphism of X (as defined in [6]).

In this paper we will give an answer for this question. Indeed, we shall prove that the set of shadowable measures is an $F_{\sigma\delta}$ set of $\mathcal{M}(X)$. In particular, the set of *invariant* shadowable measures is an $F_{\sigma\delta}$ subset of the space of invariant measures. We will also prove that the set of shadowable measures is dense in $\mathcal{M}(X)$ if and only if the shadowable points set (e.g. [1], [2], [3]) is dense in X. We will consider also a class of measures to be referred to *almost shadowable* and prove that they form together a closed subset of

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$\mathcal{M}(X)$. We also establish relationships between the shadowable points set and the set of almost shadowable measures. Namely we prove that the closure of the former set is the union of the supports of the measures in the latter set. Consequently, every almost shadowable measure can be approximated by ones with support equal to the closure of the shadowable points set. Let us state our results in a precise way.

Hereafter X will denote a compact metric space. The Borel σ-algebra of X is the σ-algebra $\mathcal{B}(X)$ generated by the open subsets of X. A *Borel probability measure* is a σ-additive measure μ defined in $\mathcal{B}(X)$ such that $\mu(X) = 1$. We denote by $\mathcal{M}(X)$ the set of all Borel probability measures of X. This set is a compact metrizable convex space and its topology is the *weak* topology* defined by the convergence $\mu_n \to \mu$ if and only if $\int \phi d\mu_n \to \int \phi d\mu$ for every continuous map $\phi : X \to \mathbb{R}$.

Let $f : X \to X$ denote a homeomorphism of a compact metric space. Given $\delta > 0$ and bi-infinite sequence $(x_n)_{n\in\mathbb{Z}}$ of points of X, say that $(x_n)_{n\in\mathbb{Z}}$ is a *δ-pseudo-orbit* if $d(f(x_n), x_{n+1}) < \delta$ for every $n \in \mathbb{Z}$. Also, if there is $y \in X$ such that $d(f^n(y), x_n) < \delta$ for every $n \in \mathbb{Z}$, then we say that $(x_n)_{n\in\mathbb{Z}}$ can be *δ-shadowed*. We say that f has the *shadowing property* [11] if for every $\epsilon > 0$ there is $\delta > 0$ such that every δ-pseudo-orbit of f can be ϵ-shadowed.

Remark 1.1. Some authors usually use non-strict inequalities in the definitions of pseudo-orbits and shadowing. The definition of the shadowing property with strict inequalities is equivalent to the classical one and allows us to prove our main results.

Next we present one of the main definitions of this work.

Definition 1.2 *([6]).* Let $f : X \to X$ be a homeomorphism of a compact metric space. We say that a Borel probability measure μ of X is *shadowable* with respect to f if for every $\epsilon > 0$ there are $\delta > 0$ and a Borelian $B \subset X$ with $\mu(B) = 1$ such that every δ-pseudo-orbit $(x_n)_{n\in\mathbb{Z}}$ *through* B (i.e., $x_0 \in B$) can be ϵ-shadowed.

Denote by $\mathcal{M}_{sh}(f)$ the set of all shadowable measures of f. Recall that a subset of a topological space is a G_δ *subset* if it is the intersection of countably many open sets. We say that it is an F_σ *subset* if it is the union of countably many closed sets. Also an $F_{\sigma\delta}$ *subset* is the intersection of countably many F_σ subsets. The $F_{\sigma\delta}$ subsets are precisely the class $\prod_3^0$ in the Borel hierarchy [4].

Now we state our first result.

Theorem 1.3. *The set of shadowable measures of a homeomorphism of a compact metric space X is an $F_{\sigma\delta}$ set of $\mathcal{M}(X)$.*

Recall that a point $x \in X$ is a *shadowable* of $f : X \to X$ if for every $\epsilon > 0$ there is $\delta > 0$ such that every δ-pseudo-orbit through x can be ϵ-shadowed. Denote by $Sh(f)$ the set of all shadowable points of f [1], [2], [3], [8]. The next result establishes a relationship between shadowable points and shadowable measures.

Theorem 1.4. *The shadowable measures of a homeomorphism of a compact metric space X are dense in $\mathcal{M}(X)$ (with respect to the weak* topology) if and only if the shadowable points are dense in X.*

From this theorem we obtain the following corollary. Recall that a metric space *has no isolated points* if no open ball reduces to a singleton. These sets are also referred to as *dense by itself*. A Borel probability measure is *non-atomic* if it has no points of positive mass.

Corollary 1.5. *Let $f : X \to X$ be a homeomorphism of a compact metric space without isolated points. If every non-atomic Borel probability measure has the shadowing property, then the shadowable points are dense in X.*

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