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Reduction of the n-dimensional static vacuum Einstein equation and generalized Schwarzschild solutions

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ABSTRACT

We consider the static vacuum Einstein spacetime when the spatial factor is conformal to a n-dimensional pseudo-Euclidean space. The most general ansatz that reduces the resulting system of partial differential equations to a system of ordinary differential equations is completely described. We obtain the entire set of solutions of the reduced system, where the classical Schwarzschild solution arises as a particular solution. In addition, we show that the Riemannian spatial factors associated to these solutions are foliated by parallel hypersurfaces of constant mean curvature.

1. Introduction

The first exact solution of the Einstein equation was obtained by Schwarzschild [31] just a few months after the publication of the Einstein's celebrated article about general relativity. The Schwarzschild solution is probably the most important solution for the Einstein equation. It is given by a static gravitational field of spherical symmetry:

$$ds^2 = \left(1 + \frac{\mu}{2r}\right)^4 (dx_1^2 + dx_2^2 + dx_3^2) - \left(\frac{1 - \frac{\mu}{2r}}{1 + \frac{\mu}{2r}}\right)^2 dt^2, \tag{1.1}$$

where μ represents the gravitating mass and $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ (see [11] for the solution given above). Through the history of the general relativity, several authors prove generalizations for the Schwarzschild solution. We can refer, for instance, to Kerr and Weyl solutions and, also, to Tangherlini [32], which supplies a higher dimensional solution for the Einstein equation. Many such exact solutions play an important role

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in the discussion of physical problems. The Schwarzschild, Kerr and Tangherlini solutions for black holes, the Friedmann solutions for cosmology, and the plane waves for gravitational radiation are some examples of applications. Even some solutions which may be thought unphysical, such like the Taub-NUT solution, have proved an invaluable guide (cf. [17,20]).

More than 100 years ago Lie [25] developed the theory of point symmetries for differential equations. Such a theory has proved to be an important tool for finding exact solutions. The possibility to reduce a partial differential equation (PDE) into an ordinary differential equation (ODE) can lead us to a great opportunity to get solutions for a given problem. Therefore, many authors have explored this subject to get new solutions for some well known problems in physics and mathematics.

In addition, this technique has been successful in problems that relate differential geometry and PDEs. For example, Tenenblat and Winternitz [33] obtained the symmetry group of the intrinsic generalized sine-Gordon and wave equations, and also their invariant solutions under the action of the group. Similarly, invariant solutions under symmetry groups for the generalized elliptic sine-Gordon equations and Laplace equations were obtained by Ferreira [13]. It is well known that such equations are related to n-dimensional Riemannian manifolds $M^n(K)$ of constant sectional curvature K isometrically immersed in a $(2n-1)$-dimensional simply connected Riemannian manifold $\overline{M}^{2n-1}(\overline{K})$, of constant sectional curvature $\overline{K}$. In this context, Barbosa, Ferreira and Tenenblat [4] showed geometric aspects of these invariant solutions by describing the geometry of the symmetry groups' orbits. Lie point symmetries also played an important role in the relation between conformally flat hypersurfaces in 4-dimensional space forms and the Lamé system of PDEs. In [10], the first author and Tenenblat obtained the symmetry group of such system and described its invariant solutions as well as its geometric aspects. As a consequence, a new class of conformally flat hypersurfaces was presented.

A common feature of the results mentioned above are the solutions invariant under a symmetry group which is given by translations. In all cases, these solutions were explicitly described and interesting geometric properties were presented. In this context, solutions of important problems were obtained under a special substitution given by a function invariant under translations. For example, Barbosa, Tenenblat and Pina [5] use the invariance under translations to get examples of semi-Riemannian gradient steady Ricci solitons. Sousa and Pina [23] also use this invariance to provide new examples of Einstein warped products, where the spatial factor is conformal to an n-dimensional pseudo-Euclidean space invariant under the action of a translation group. In particular, they obtained a family of static vacuum spacetimes. With the same geometric invariant, the second author and Pina [21] obtained a class of static vacuum spacetimes with non-null cosmological constant. The second author and Tenenblat [22] also use the translation group to get new examples of Yamabe solitons.

Roughly, an *ansatz* is a substitution of variables which reduces the original PDE to an ODE or another PDE with less independent variables. The Lie's theory of symmetry groups for differential equations provides *ansatz* by considering invariant functions under the action of a symmetry group using the algebra of infinitesimal generators. Another method for finding *ansatze*, that does not make use of group theory, was known as direct method and was introduced in a systematic approach by Clarkson and Kruskal [9], where the Boussinesq equation was considered. Later, Levi and Winternitz [24] gave an interpretation by means of group theory to the method developed by Clarkson and Kruskal. They show that it was related to the method of developed by Bluman and Cole [7], also known as nonclassical symmetries. A generalization of the direct method was purposed by Galaktionov [15], where the *ansatz* is used to reduce a PDE to a system of ODE, whereas Olver [28] proved that a PDE admits a direct reduction to a system of ODE if and only if there is a compatibility of the equation with a differential constraint. Although the direct method has been introduced for two-dimensional equations, it was also successful when multiple variables were considered, as we can see, for example, for the Schrödinger equations [35] and for the nonlinear wave equations [14]. A formal definition of an *ansatz* was given by Zhdanov, Tsyfra and Popovych [36], where a relation between direct reductions and conditional symmetries, introduced by [7], is given.

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