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Fast and provable algorithms for spectrally sparse signal reconstruction via low-rank Hankel matrix completion

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ABSTRACT

A spectrally sparse signal of order r is a mixture of r damped or undamped complex sinusoids. This paper investigates the problem of reconstructing spectrally sparse signals from a random subset of n regular time domain samples, which can be reformulated as a low rank Hankel matrix completion problem. We introduce an iterative hard thresholding (IHT) algorithm and a fast iterative hard thresholding (FIHT) algorithm for efficient reconstruction of spectrally sparse signals via low rank Hankel matrix completion. Theoretical recovery guarantees have been established for FIHT, showing that $O(r^2 \log^2(n))$ number of samples are sufficient for exact recovery with high probability. Empirical performance comparisons establish significant computational advantages for IHT and FIHT. In particular, numerical simulations on 3D arrays demonstrate the capability of FIHT on handling large and high-dimensional real data.

1. Introduction

Spectrally sparse signals arise frequently from various applications, ranging from magnetic resonance imaging [23], fluorescence microscopy [28], radar imaging [24], nuclear magnetic resonance (NMR) spectroscopy [25], to analog-to-digital conversion [32]. For ease of presentation, consider a one-dimensional (1-D) signal which is a weighted superposition of r complex sinusoids with or without damping factors

$$x(t) = \sum_{k=1}^{r} d_k e^{(2\pi \imath f_k - \tau_k)t}, \tag{1}$$

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where r is the model order, $\imath = \sqrt{-1}$, $f_k \in [0, 1)$ is the normalized frequency with respect to the signal bandwidth, $d_k \in \mathbb{C}$ is the corresponding complex amplitude, and $\tau_k \geq 0$ is the damping factor. Let $\boldsymbol{x} = [x_0, \quad \cdots, \quad x_{n-1}]^T \in \mathbb{C}^n$ be the discrete samples of $x(t)$ at $t \in \{0, \cdots, n-1\}$; that is,

$$\boldsymbol{x} = [x(0), \quad \cdots, \quad x(n-1)]^T. \tag{2}$$

Under many circumstances of practical interests, $x(t)$ can only be sampled at a subset of times in $\{0, \cdots, n-1\}$ due to costly experiments [25], hardware limitation [32], or other inevitable reasons. Consequently only partial entries of $\boldsymbol{x}$ are known. Thus we need to reconstruct $\boldsymbol{x}$ from its observed entries in these applications. Let $\Omega \subset \{0, \cdots, n-1\}$ with $|\Omega| = m$ be the collection of indices of the observed entries. The reconstruction problem can be expressed as

$$\text{find} \quad \boldsymbol{x} \quad \text{subject to} \quad \mathcal{P}_\Omega(\boldsymbol{x}) = \sum_{a \in \Omega} x_a \boldsymbol{e}_a, \tag{3}$$

where $\boldsymbol{e}_a$ is the a-th canonical basis of $\mathbb{C}^n$, and $\mathcal{P}_\Omega$ is a projection operator defined as

$$\mathcal{P}_\Omega(\boldsymbol{z}) = \sum_{a \in \Omega} \langle \boldsymbol{z}, \boldsymbol{e}_a \rangle \boldsymbol{e}_a.$$

Generally it is impossible to reconstruct a vector from its partial entries since the unknown entries can take any values without violating the equality constraint in (3). However, the theory of compressed sensing [11,16] and matrix completion [10,27] suggests that signals with inherent simple structures can be uniquely determined from a number of measurements that is less than the size of the signal. In a spectrally sparse signal, the number of unknowns is at most $3r$, which is smaller than the length of the signal if $r \ll n$. Therefore it is possible to reconstruct $\boldsymbol{x}$ from $\mathcal{P}_\Omega \boldsymbol{x}$.

This paper exploits the low rank structure of the Hankel matrix constructed from $\boldsymbol{x}$. Let $\mathcal{H}$ be a linear operator which maps a vector $\boldsymbol{z} \in \mathbb{C}^n$ to a Hankel matrix $\mathcal{H}\boldsymbol{z} \in \mathbb{C}^{n_1 \times n_2}$ with $n_1 + n_2 = n + 1$ as follows

$$[\mathcal{H}\boldsymbol{z}]_{ij} = z_{i+j}, \quad \forall\, i \in \{0, \ldots, n_1 - 1\},\ j \in \{0, \ldots, n_2 - 1\},$$

where vectors and matrices are indexed starting with zero, and $[\cdot]_{ij}$ denotes the (i, j)-th entry of a matrix. Define $y_k = e^{(2\pi\imath f_k - \tau_k)}$ for $k = 1, \ldots, r$. Since $\boldsymbol{x}$ is a spectrally sparse signal, the Hankel matrix $\mathcal{H}\boldsymbol{x}$ admits a Vandermonde decomposition

$$\mathcal{H}\boldsymbol{x} = \boldsymbol{E}_L \boldsymbol{D} \boldsymbol{E}_R^T,$$

where

$$\boldsymbol{E}_L = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ y_1 & y_2 & \cdots & y_r \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{n_1-1} & y_2^{n_1-1} & \cdots & y_r^{n_1-1} \end{bmatrix}, \quad \boldsymbol{E}_R = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ y_1 & y_2 & \cdots & y_r \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{n_2-1} & y_2^{n_2-1} & \cdots & y_r^{n_2-1} \end{bmatrix}$$

and $\boldsymbol{D}$ is a diagonal matrix whose diagonal entries are $d_1, \ldots, d_r$. If all y_k's are distinct and $r \leq \min(n_1, n_2)$, $\boldsymbol{E}_L$ and $\boldsymbol{E}_R$ are both full rank matrices. Therefore $\text{rank}(\mathcal{H}\boldsymbol{x}) = r$ when all d_k's are non-zeros. Since $\mathcal{H}$ is injective, the reconstruction of $\boldsymbol{x}$ from $\mathcal{P}_\Omega(\boldsymbol{x})$ is equivalent to the reconstruction of $\mathcal{H}\boldsymbol{x}$ from partial revealed anti-diagonals that corresponds to the known entries of $\boldsymbol{x}$. With a slight abuse of notation we also use $\mathcal{P}_\Omega$ to denote the projection of a matrix $\boldsymbol{Z} \in \mathbb{C}^{n_1 \times n_2}$ onto the subspace determined by a subset of an orthonormal basis of Hankel matrices; that is,