Appl. Comput. Harmon. Anal. • • • (• • • •) • • •–• • •

Contents lists available at ScienceDirect

Applied and Computational Harmonic Analysis

www.elsevier.com/locate/acha

Rotation invariant, Riesz bases of directional wavelets

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A R T I C L E   I N F O

Article history:
Received 18 July 2016
Received in revised form 5 February 2017
Accepted 11 April 2017
Available online xxxx
Communicated by Bruno Torresani

Keywords:
Directional wavelets
Riesz basis

A B S T R A C T

This article addresses the issue of designing bases for $L^2(\mathbb{R}^2)$ that are generated by translations, rotations and dilations of a single mother wavelet ψ. We show how this construction can be simplified by setting an odd number of directions and by choosing properly the phase of the Fourier transform of ψ. A large part of the article is devoted to the proof of theorems that give sufficient conditions for ψ to generate a Riesz sequence and a Riesz basis for $L^2(\mathbb{R}^2)$. An example of Riesz sequence whose restriction to each scale is orthonormal is set. Theoretical results are confirmed by numerical experiments where a discrete directional wavelet transform is introduced.

1. Introduction

For more than two decades, the wavelet transform has been acknowledged as a very powerful tool in image processing. It is still widely used in many areas such as restoration and classification, but its efficiency is especially renowned for its applications in image compression through JPEG2000 [19], ICER [17] and CCSDS [6] standards.

As wavelets were originally designed for 1D signals, they have been extended to 2D images via tensor product. Since then, many transforms have been introduced as alternatives to the separable wavelet transform in order to represent more efficiently directional structures. The most famous examples include hexagonal wavelets [28,8], steerable pyramids [29], directional wavelets [1], dual-tree complex wavelets [20,27], ridgelets [4], curvelets [5,3], contourlets [9], shearlets [14], wavelets with composite dilations [15,16] and related constructions such that Haar-type wavelets [18] or crystallographic wavelets [2,21,22]. All these transforms amount to express the image in a frame. But, except for orthonormal ridgelets [10] that are designed for images with straight edges, up to now only transforms which rely on a filterbank tree are known to be possibly non-redundant and can therefore be used in compression [30,26,11,15,16,18,12,13,2,31,22]. Filterbanks have well known digital advantages, but suffer the drawback that the decimation generates aliasing on the bases functions as it is shown in Fig. 1-(a) that is extracted from [13], and it produces

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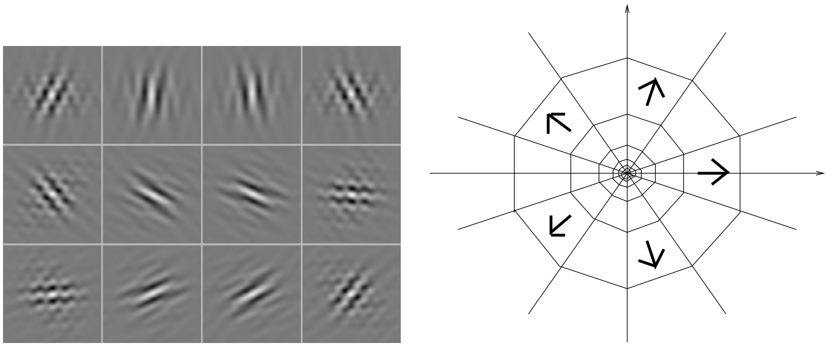
http://dx.doi.org/10.1016/j.acha.2017.04.001
1063-5203/



(a) (b)

Fig. 1. (a) Directional wavelets generated by a filterbank tree are aliased [13]. (b) The proposed directional wavelet basis has an odd number N of directions, and is invariant under rotation of angle $\frac{2\pi}{N}$.

therefore artifacts on the compressed image. Thus, the design of directional bases that have all the required properties remains an open problem.

Notice that we did not mention adaptive multiscale transforms such as bandlets [23]. The second generation bandlets are non-redundant [24], but the separable wavelet basis used in their design could be replaced by any directional wavelet basis in order to get better approximations.

In this article, we focus on Antoine and Murenzi's directional wavelets [1,25]. This sequence stands out from most of the other examples quoted above thanks to the simplicity of its construction. It is generated by a single wavelet ψ in the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ to which a composition of translations, dilations and rotations is applied. Denote by R the rotation, by Γ the regular lattice along which the wavelet is shifted, by Γ^* its dual lattice and put $\psi_{j,k,\gamma} = 2^j \psi(2^j R^k \cdot - \gamma)$. It is well known that if the function $\sum_{j\in\mathbb{Z}} \sum_{k\in\{0,\cdots,N-1\}} |\widehat{\psi}(2^j R^k \cdot)|^2$ is positive and if $\operatorname{supp} \widehat{\psi}(\cdot) \cap \operatorname{supp} \widehat{\psi}(\cdot - \gamma) = \emptyset$, for all $\gamma \in \Gamma^*$, then $(\psi_{j,k,\gamma})_{j\in\mathbb{Z},k\in\{0,\cdots,N-1\},\gamma\in\Gamma}$ is a frame for $L^2(\mathbb{R}^2)$.

It is generally believed that one cannot obtain bases for $L^2(\mathbb{R}^2)$ with this approach. This article aims to contradict this idea. More precisely, we show that, if the number of rotations N is odd, the phase of the Fourier transform $\widehat{\psi}$ is well chosen and $\widehat{\psi}$ satisfies some specific inequalities, then the lattice Γ can be changed into a larger one so that $(\psi_{j,k,\gamma})_{(j,k,\gamma)\in\mathbb{Z}\times\{0,\cdots,N-1\}\times\Gamma}$ becomes surprisingly a Riesz sequence or possibly a Riesz basis for $L^2(\mathbb{R}^2)$.

2. Notations and preliminaries

2.1. Notations

All along this article, we will use the following notations.

- $\mu = (2\pi, 0)$, $\nu = \left(\pi, 3\pi \tan\left(\frac{\pi}{2N}\right)\right)$ and $\eta = \left(\pi, \pi \tan\left(\frac{\pi}{2N}\right)\right)$—see Fig. 2.
- $\Gamma = \mathbb{Z}\left(1, \frac{-1}{3\tan(\pi/2N)}\right) + \mathbb{Z}\left(0, \frac{2}{3\tan(\pi/2N)}\right)$.
- The dual lattice of Γ is therefore

$$\Gamma^* = \{\gamma \in \mathbb{R}^2 \,:\, \gamma \cdot \lambda \in 2\pi\mathbb{Z}, \forall \lambda \in \Gamma\} = \mathbb{Z}\mu + \mathbb{Z}\nu.$$

- For all $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, we denote by $\tilde{\xi} = (-\xi_1, \xi_2)$ its symmetrical point with respect to the ordinate axis.
- R is the rotation of angle $\frac{(N-1)\pi}{N}$ about 0. Observe that $R\eta = \tilde{\eta}$.