J. Math. Anal. Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Thin sets of constant width

Elisabetta Maluta [a,*,1], David Yost [b]

[a] *Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano, MI, Italy*
[b] *Centre for Informatics and Applied Optimisation, Federation University, PO Box 663, Ballarat 3353, Australia*

A R T I C L E I N F O

Article history:
Received 25 May 2018
Available online xxxx
Submitted by T. Domínguez Benavides

Keywords:
Constant width
Diametrically complete set
Reflexive space
Normal structure

A B S T R A C T

We prove that every Banach space which admits an unconditional basis can be renormed to contain a constant width set with empty interior, thus guaranteeing, for the first time, existence of such sets in a reflexive space. In the isometric case we prove that normal structure is characterized by the property that the class of diametrically complete sets and the class of sets with constant radius from the boundary coincide.

0. Introduction

Sets of constant width, i.e. sets such that the distance between any pair of parallel hyperplanes supporting the set is constant, were first studied (at least in the Euclidean plane) by Euler [10, paragraph 5] in the eighteenth century; the most famous non-trivial example is probably the Reuleaux triangle. At the beginning of the last century, Meissner [20] noticed that sets of constant width have the property that adding any point to the set produces a set of strictly larger diameter. The property is now called diametrical completeness.

The two properties coincide in 2-dimensional and in Euclidean spaces, but they fail to coincide in some 3-dimensional Minkowski spaces, as first noted in [9]. Later [28, Corollary 22] it was shown that most norms on $\mathbb{R}^d$ (in the sense of Baire category) give rise to diametrically complete sets which are not of constant width.

In finite dimensional spaces, both properties have been widely investigated, and classical results can be found for example in [19], [28] and the references therein.

* Corresponding author.
E-mail addresses: elisabetta.maluta@polimi.it (E. Maluta), d.yost@federation.edu.au (D. Yost).

[1] Research of the first author was partially supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) of Italy.

https://doi.org/10.1016/j.jmaa.2018.09.053
0022-247X/

In infinite dimensional spaces, the concept was slower to take hold. It is easy to verify, and it seems futile to attribute this folklore result to anyone particular, that the positive part of the unit ball of c_0 is a set which has both constant width and empty interior, strongly contrasting with the intuitive idea of constant width sets as being "plump" sets, which is familiar from the finite dimensional setting.

In 1975, Holmes, Scranton and Ward [12] proved that if M is an M-ideal in a Banach space X, then the sets of best approximation $P_M(x)\colon = \{y\colon y \in M, \|x - y\| = d(x, M)\}$ are constant width sets in M. In 1979, Behrends [2, Proposition 6.6] gave a more natural proof of this. The simplest non-trivial example of an M-ideal is of course the subspace c_0 in ℓ_∞. If $x \in \ell_\infty$ is the constant sequence $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots)$, then the set of best approximations $P_M(x)$ is just the positive part of the unit ball of c_0. So this example can be interpreted as a byproduct of the study of M-ideals. However, it is easy to check this example without knowing what M-ideals are. None of these authors used the constant width terminology, and it was several years before interest picked up.

Behrends and Harmand [3] defined a class of closed convex sets called pseudoballs (see §2). It follows from the proof of [3, Theorem 3.4] that every pseudoball is a set of constant width, but again they did not use this terminology. They did show that any Banach space containing a non-trivial pseudoball necessarily contains an almost isometric copy of c_0. So the example from the previous paragraph is somehow ubiquitous. The first serious study of sets of constant width in infinite dimensions was made by Payá and Rodriguez-Palacios [25], still motivated by the study of M-ideals. Some scattered results appeared in the following twenty years. Then in 2006, Moreno, Papini and Phelps published two papers [23,24] containing a thorough study of both constant width and diametrically complete sets in the spaces $C(K)$, and an exhaustive report of the results obtained until that time.

In a d-dimensional Banach space, it is almost obvious that a set of constant width w must have affine dimension d, and thus non-empty interior. Less obvious is that it must contain a ball of radius $w/(d+1)$. This was proved by Leichtweiss [13, Theorems 3 and 4], who also showed that this estimate is best possible, as exemplified by $X = \mathbb{R}^d$ with the norm $\|(x_1, \cdots, x_d)\| = \sum_{i=1}^d |x_i| + |\sum_{i=1}^d x_i|$, and C being the convex hull of $\{0, e_1, e_2, \cdots, e_d\}$. The dual inequality for balls containing a given set is due to Bohnenblust [4]. For other recent work on this topic, see [27, Theorem 2], [22, Theorem 2], [5, (5)], and the references therein.

This suggests that infinite dimensional Banach spaces may contain sets of constant width with empty interior, and we have seen that this is the case in c_0. On the other hand, ℓ_1 does not contain any sets of constant width other than balls. To see this, note first that the unit ball of c_0 is clearly the absolutely convex hull of a proper face, which implies by [14, Proposition 8] that every pseudolinear map on c_0 is linear; we refer to [14] for appropriate definitions, since we will not need them again here. Then [14, Theorem 7] tells us that ℓ_1 does not contain any nontrivial weak* compact set of constant width. The conclusion then follows from [25, Corollary 1.2], which asserts that any set of constant width in a dual space must be weak* compact. However the existence of sets of constant width with empty interior in some reflexive Banach spaces has until now remained an open question.

Research then moved toward the formally weaker question, of looking for diametrically complete sets with empty interior. The first such attempts in Banach spaces looked for isometric results, in particular trying to characterize Banach spaces containing such sets. Though a full characterization has not yet been obtained, Maluta and Papini [18] proved existence of such sets in many reflexive Banach spaces, including spaces with good geometric properties [17].

More recently, attention has turned to the isomorphic setting, where existence of sets with the requested properties is guaranteed up to a suitable renorming of the space. Specifically, it is shown in [6] that many reflexive Banach spaces contain diametrically complete sets with empty interior. We strengthen this here, proving also that many reflexive Banach spaces contain constant width sets with empty interior.

Organization of the paper is as follows.

In Section 1 we present the basic definitions and some known results needed in the paper.

Please cite this article in press as: E. Maluta, D. Yost, Thin sets of constant width, J. Math. Anal. Appl. (2018), https://doi.org/10.1016/j.jmaa.2018.09.053