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Exact recovery low-rank matrix via transformed affine matrix rank minimization

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ABSTRACT

The goal of affine matrix rank minimization problem is to reconstruct a low-rank or approximately low-rank matrix under linear constraints. In general, this problem is combinatorial and NP-hard. In this paper, a nonconvex fraction function is studied to approximate the rank of a matrix and translate this NP-hard problem into a transformed affine matrix rank minimization problem. The equivalence between these two problems is established, and we proved that the uniqueness of the global minimizer of transformed affine matrix rank minimization problem also solves affine matrix rank minimization problem if some conditions are satisfied. Moreover, we also proved that the optimal solution to the transformed affine matrix rank minimization problem can be approximately obtained by solving its regularization problem for some proper smaller $\lambda > 0$. Lastly, the DC algorithm is utilized to solve the regularization transformed affine matrix rank minimization problem and the numerical experiments on image inpainting problems show that our method performs effectively in recovering low-rank images compared with some state-of-art algorithms.

1. Introduction

The goal of affine matrix rank minimization (AMRM) problem is to reconstruct a low-rank or approximately low-rank matrix that satisfies a given system of linear equality constraints. In mathematics, it can be described as the following minimization problem

$$(\text{AMRM}) \qquad \min_{X\in\mathbb{R}^{m\times n}} \ \text{rank}(X) \quad s.t. \quad \mathcal{A}(X) = b, \tag{1}$$

where $\mathcal{A}: \mathbb{R}^{m\times n} \mapsto \mathbb{R}^d$ is the linear map and the vector $b \in \mathbb{R}^d$. Without loss of generality, we assume $m \leq n$. Many applications arising in various areas can be captured by solving the problem (AMRM), for instance, the network localization [1], the minimum order system and low-dimensional Euclidean embedding in control theory [2,3], the collaborative filtering in recommender systems [4,5], and so on. One important special case of the problem (AMRM) is the matrix completion (MC) problem [4]

$$(\text{MC}) \qquad \min_{X\in\mathbb{R}^{m\times n}} \ \text{rank}(X) \quad s.t. \quad X_{i,j} = M_{i,j}, \quad (i,j) \in \Omega. \tag{2}$$

This completion problem has been applied in the famous Netflix problem [6], image inpainting problem [7] and machine learning [8,9]. In general, however, the problem (AMRM) is a challenging non-convex optimization problem and is known as NP-hard [10] due to the combinational nature of the rank function.

Among the numerous substitution models, the nuclear-norm affine matrix rank minimization (NAMRM) problem has been considered as the most popular alternative [3,4,11–13]:

$$(\text{NAMRM}) \qquad \min_{X\in\mathbb{R}^{m\times n}} \ \|X\|_* \quad s.t. \quad \mathcal{A}(X) = b. \tag{3}$$

where $\|X\|_* = \sum_{i=1}^{m}\sigma_i(X)$ is the nuclear-norm of the matrix $X \in \mathbb{R}^{m\times n}$. Recht et al. in [10] have show that if a certain restricted isometry property (RIP) holds for the linear transformation defining the constraints, the minimum rank solution of problem (AMRM) can be recovered by solving the problem (NAMRM). In addition, some popular methods, including singular value thresholding algorithm [14], proximal gradient algorithm [15] and accelerated proximal gradient algorithm [16], are proposed to solve its

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regularization (or Lagrangian) version:

$$\text{(RNAMRM)} \qquad \min_{X\in\mathbb{R}^{m\times n}} \left\{ \|\mathcal{A}(X)-b\|_2^2+\lambda\|X\|_* \right\}, \tag{4}$$

where $\lambda > 0$ is the regularization parameter can be selected to guarantee that solutions of the problem (NAMRM) and (RNAMRM) are same [17]. However, these algorithms tend to have biased estimation by shrinking all the singular values toward zero simultaneously, and sometimes results in over-penalization in the regularization problem (RNAMRM) as the ℓ_1-norm in compressive sensing. Moreover, with the recent development of non-convex relaxation approach in sparse signal recovery problems, many researchers have shown that using a non-convex surrogate function to approximate the ℓ_0-norm is a better choice than using the ℓ_1-norm. This brings our attention back to the non-convex surrogate functions of the rank function.

In this paper, a continuous promoting low-rank non-convex function

$$P_a(X)=\sum_{i=1}^{m}\rho_a(\sigma_i(X))=\sum_{i=1}^{m}\frac{a\sigma_i(X)}{a\sigma_i(X)+1} \tag{5}$$

in terms of the singular values of matrix X is considered to substitute the rank function rank(X) in the problem (AMRM), where the non-convex function

$$\rho_a(t)=\frac{a|t|}{a|t|+1} \quad (a>0) \tag{6}$$

is the fraction function. It is to see clearly that, with the change of parameter $a > 0$, the non-convex function $P_a(X)$ approximates the rank of matrix X:

$$\lim_{a\to+\infty}P_a(X)=\lim_{a\to+\infty}\sum_{i=1}^{m}\frac{a\sigma_i(X)}{a\sigma_i(X)+1}\approx\begin{cases}0, & \text{if } \sigma_i(X)=0;\\ \text{rank}(X), & \text{if } \sigma_i(X)>0.\end{cases} \tag{7}$$

By this transformation, the NP-hard problem (AMRM) could be relaxed into the following matrix rank minimization problem with a continuous non-convex penalty, namely, transformed affine matrix rank minimization (TrAMRM) problem:

$$\text{(TrAMRM)} \qquad \min_{X\in\mathbb{R}^{m\times n}} P_a(X) \quad s.t. \quad \mathcal{A}(X)=b, \tag{8}$$

where the non-convex surrogate function $P_a(X)$ in terms of the singular values of matrix X is defined in (5). Unfortunately, although we relax the NP-hard problem (AMRM) into a continuous problem (TrAMRM), this relax problem is still computationally harder to solve due to the non-convex nature of the function $P_a(X)$, in fact it is also NP-hard. Frequently, we consider its regularization version:

$$\text{(RTrAMRM)} \qquad \min_{X\in\mathbb{R}^{m\times n}} \left\{ \|\mathcal{A}(X)-b\|_2^2+\lambda P_a(X) \right\}, \tag{9}$$

where $\lambda > 0$ is the regularization parameter. Unlike the convex optimal theory, there are no parameters $\lambda > 0$ such that the solution to the regularization problem (RTrAMRM) also solves the constrained problem (TrAMRM). However, as the unconstrained form, the problem (RNuAMRM) may possess much more algorithmic advantages. Moreover, we also proved that the optimal solution to the problem (TrAMRM) can be approximately obtained by solving the problem (RTrAMRM) for some proper smaller $\lambda > 0$.

The rest of this paper is organized as follows. Some notions and preliminary results that are used in this paper are given in Section 2. In Section 3, the equivalence of the problem (TrAMRM) and (AMRM) is established. Moreover, we proved that the optimal solution to the problem (TrAMRM) can be approximately obtained by solving the problem (RTrAMRM) for some proper smaller $\lambda > 0$. In Section 4, the DC algorithm is utilized to solve the problem (RTrAMRM) and the numerical results of the numerical experiments on image inpainting problems are demonstrated in Section 5. Finally, we give some concluding remarks in Section 6.

2. Preliminaries

In this section, we give some notions and preliminary results that are used in this paper.

2.1. Some notions

The linear map $\mathcal{A}:\mathbb{R}^{m\times n}\mapsto\mathbb{R}^d$ determined by d matrices $A_1, A_2, \ldots, A_d \in \mathbb{R}^{m\times n}$ can be expressed as $\mathcal{A}(X)=(\langle A_1,X\rangle, \langle A_2,X\rangle,\ldots,\langle A_d,X\rangle)^\top\in\mathbb{R}^d$. Let $A=(vec(A_1),vec(A_2),\ldots,vec(A_d))^T\in\mathbb{R}^{d\times mn}$ and $x=vec(X)\in\mathbb{R}^{mn}$. Then we can get that $\mathcal{A}(X)=Ax$. The standard inner product of matrices $X\in\mathbb{R}^{m\times n}$ and $Y\in\mathbb{R}^{m\times n}$ is denoted by $\langle X, Y\rangle$, and $\langle X,Y\rangle=\text{tr}(Y^TX)$. The $\mathcal{A}^*$ denotes the adjoint of $\mathcal{A}$, and for any $y\in\mathbb{R}^d$, $\mathcal{A}^*(y)=\sum_{i=1}^d y_iA_i$. The singular value decomposition (SVD) of matrix X is $X=U\Sigma V^T$, where U is an $m\times m$ unitary matrix, V is an $n\times n$ unitary matrix and $\Sigma=\text{Diag}(\sigma(X))\in\mathbb{R}^{m\times n}$ is a diagonal matrix. The vector $\sigma(X)$: $\sigma_1(X)\ge\sigma_2(X)\ge\cdots\ge\sigma_m(X)$ arranged in descending order represents the singular values vector of matrix X, and $\sigma_i(X)$ denotes the ith largest singular value of matrix X for $i=1,2,\ldots,m$.

2.2. Some useful results

Lemma 1. *(see [10]) Let M and N be matrices of the same dimensions. Then there exist matrices N_1 and N_2 such that*
(1) $N=N_1+N_2$;
(2) $\text{rank}(N_1)\le 2\text{rank}(M)$;
(3) $MN_2{}^T=0$ and $M^TN_2=0$;
(4) $\langle N1_1,N_2\rangle=0$.

By Lemma 1, we can derive the following important corollary.

Corollary 1. *Let X^* and X_0 be the optimal solutions to the problem (TrAMRM) and (AMRM), respectively. If we set $R=X^*-X_0$, then there exist matrices R_0 and R_c such that*
(1) $R=R_0+R_c$;
(2) $\text{rank}(R_0)\le 2\text{rank}(X_0)$;
(3) $X_0R_c{}^T=0$, $X_0{}^TR_c=0$ and $\langle R_0,R_c\rangle=0$.

Lemma 2. *Let M and N be matrices of the same dimensions. If $MN^T=0$ and $M^TN=0$, then $P_a(M+N)=P_a(M)+P_a(N)$.*

Proof. Consider the SVDs of the matrices M and N:

$$M=U_M\begin{bmatrix}\Sigma_M & 0\\ 0 & 0\end{bmatrix}V_M{}^T, \quad N=U_N\begin{bmatrix}\Sigma_N & 0\\ 0 & 0\end{bmatrix}V_N{}^T. \tag{10}$$

Since U_M and U_N are left-invertible, the condition $MN^T=0$ implies that $V_M{}^TV_N=0$. Similarly, $M^TN=0$ implies that $U_M{}^TU_N=0$. Thus, the following is a valid SVD for $M+N$,

$$M+N=\begin{bmatrix}U_M & U_N\end{bmatrix}\begin{bmatrix}\Sigma_M & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & \Sigma_N & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}V_M & V_N\end{bmatrix}^T. \tag{11}$$

This shows that the singular values of $M+N$ are equal to the union (with repetition) of the singular values of M and N. Hence, $P_a(M+N)=P_a(M)+P_a(N)$. This completes the proof. □

Combing Corollary 1 and Lemma 2, we can get the following corollary.

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