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Around Wilson's theorem

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ABSTRACT

We study the series s(n, x) which is the sum for k from 1 to n of the square of the sine of the product x Gamma(k)/k, where x is a variable. By Wilson's theorem we show that the integer part of s(n, x) for x = Pi/2 is the number of primes less or equal to n and we get a similar formula for x a rational multiple of Pi. We show that for almost all x in the Lebesgue measure s(n, x) is equivalent to n/2 when n tends to infinity, while for almost all x in the Baire sense, 1/2 is a limit point of the ratio of s(n, x) to the number of primes less or equal to n.

1. Introduction

Let $\Pi(n)$ be the number of primes $p \leq n$. A slight improvement on a formula (see Fig. 1) of Willans [3] gives a simple formula for $\Pi(n)$ as the integer part of the sum

$$\sum_{k=1}^{n} \sin^2\left(\frac{\pi\Gamma(k)}{2k}\right) \tag{1}$$

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Then $$H(x) = \frac{\sin^2 \pi \dfrac{\{(x-1)!\}^2}{x}}{\sin^2 \dfrac{\pi}{x}} \begin{cases} =1 \text{ for } x \text{ prime,} \\ =0 \text{ for } x \text{ composite.} \end{cases}$$

It follows that

$$\pi(m) = \sum_{x=2}^{m} H(x) \quad \text{for } m = 2, 3, \ldots$$

Fig. 1. The formula of Willans.

When one tries to compute naively the right hand side one finds that it requires an increasing precision on the numerical value of the number π whose first 2500 decimals are needed to compute $\Pi(n)$ for n of the order of a thousand. F. Villegas suggested to replace π by a variable and analyze the dependence on x in the above sequence. Thus for $n > 1$ an integer and $x \in \mathbb{R}$, let

$$s(n, x) := \sum_{k=1}^{n} \sin^2 \left(\frac{x\Gamma(k)}{k} \right) \tag{2}$$

We shall show below that the dependence on $x \in \mathbb{R}$ is quite interesting inasmuch as the terms of the sum (1) are independent random variables when suitably understood as functions on an almost periodic compactification G of $\mathbb{R}$. This easily gives by the strong law of large numbers that for almost all $x \in \mathbb{R}$ in the sense of the Lebesgue measure one has when $n \to \infty$ that $s(n, x) \sim \frac{n}{2}$. The interesting fact is that for the other natural notion of "generic" real number, namely the one provided by the Baire theory of dense countable intersections of open sets, it is a totally different behavior of the sequence $s(n, x)$ which is generic: we show in Theorem 4.1 that for generic $x \in \mathbb{R}$, the quotients $\frac{s(n,x)}{\Pi(n)}$ get arbitrarily close to $\frac{1}{2}$, *i.e.* $\frac{1}{2}$ is a limit point of the sequence

$$\frac{1}{2} \in \lim_{n \to \infty} \frac{s(n, x)}{\Pi(n)}.$$

Generically this sequence will also have ∞ as a limit point and will oscillate wildly. But for rational multiples of π the sequence $s(n, x)$ behaves like the product of $\Pi(n)$ by the rational number[1] $\frac{1}{2} - \frac{\mu(b)}{2\phi(b)}$ which only depends upon the denominator $b > 1$ of the irreducible fraction $x = \frac{a}{b}\pi$ as a multiple of π (see Proposition 3.1).

2. $\Pi(n)$ and sum of squared sines

We start with the following variant of the formulas of Willans [3].

Proposition 2.1. *Let $n > 1$ be an integer then $\Pi(n)$ is the integer part of $s(n, \frac{\pi}{2})$.*

[1] μ is the Möebius function and ϕ the Euler totient function.

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