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Zero-density estimate of L-functions for cusp forms

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ABSTRACT

Let $f(z)$ be a Maass cusp form for the full modular group $SL_2(\mathbb{Z})$, and let $L(s, f)$ be the corresponding automorphic L-function associated to f. In this paper, we give a new result concerning zero-density estimate of $L(s, f)$ in the range $3/4 < \sigma < 1$. The proof follows the classical line of Halász–Montgomery–Jutila-Ivić.

1. Introduction

Zero-density estimates for the Riemann–Zeta function $\zeta(s)$ or Dirichlet L-functions $L(s, \chi)$ are significant topics in classical analytic number theory which has important applications in many problems concerning distribution of primes. Denote by $N(\sigma, T)$ the number of zeros of $\zeta(s)$ in the region $\{\sigma \leq \Re s \leq 1, |t| \leq T\}$. Zero-density estimates seek bounds of the form $N(\sigma, T) \ll T^{A(\sigma)(1-\sigma)} \log^c T$, and $A(\sigma) = 2$ is the zero-density

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conjecture. This conjecture was proved to be true when σ is near 1 (see, for example, [1], [12]). So far the best uniform bound in the range $\sigma \in [1/2, 1]$ is $A(\sigma) \leq 12/5$, this can be seen, for example, from the classical result of Ingham ([7]): $A(\sigma) \leq 3/(2-\sigma)$ for $1/2 \leq \sigma \leq 1$, and result of Huxley ([4]): $A(\sigma) \leq 3/(3\sigma - 1)$ for $3/4 \leq \sigma \leq 1$. In [9], Ivić, gave the following bound: $N(\sigma, T) \ll T^{A(\sigma)(1-\sigma)+\varepsilon}$ with

$$A(\sigma) = \begin{cases} \frac{3}{7\sigma-4} & \text{for } \frac{3}{4} \leq \sigma \leq \frac{10}{13} \\ \frac{9}{8\sigma-2} & \text{for } \frac{10}{13} \leq \sigma \leq 1. \end{cases} \tag{1.1}$$

Several techniques have been developed in obtaining zero-density estimates for $\zeta(s)$, and modifications of the method have been applied to obtain similar results concerning zero-density estimates of $L(s, \chi)$ (see, for example, [5], [6]).

Motivated by the classical zero-density problem, many authors considered zero-density problem for L-functions associated to a holomorphic or Maass cusp form f for the full modular group $SL_2(\mathbb{Z})$. Here the L-function $L(s, f)$ is defined as

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \lambda_f(p)p^{-s} + p^{-2s}\right)^{-1}, \quad \Re(s) > 1,$$

where $\lambda_f(n) \in \mathbb{R}$ $(n = 1,\ 2,\ \ldots)$ are eigenvalues of the Hecke operators $T(n)$ (that is, $T(n)f = \lambda(n)f$) and is normalized so that $\lambda_f(1) = 1$ (see [11] and [2]). As is well known that the above series converges absolutely for $\Re(s) > 1$, and $L(s, f)$ can be continued analytically to the whole complex plane. The GRH predicts that the non-trivial zeros of $L(s, f)$ all lie on the critical line $\Re(s) = 1/2$, and the zero-density conjecture states $N_f(\sigma, T) \ll T^{2(1-\sigma)} \log^c T$ or $N_f(\sigma, T) \ll T^{2(1-\sigma)+\varepsilon}$ for the log-free form, where $c > 0$ and $\varepsilon > 0$ is arbitrary. Here $N_f(\sigma, T)$ denote the number of zeros of $L(s, f)$ in the region $\{\sigma \leq \beta \leq 1, |\gamma| \leq T\}$. When f is a holomorphic cusp form, Ivić considered this problem and showed in [10] that $N_f(\sigma, T) \ll T^{A(\sigma)(1-\sigma)+\varepsilon}$, where

$$A(\sigma) = \begin{cases} \frac{4}{3-2\sigma} & \text{for } \frac{1}{2} \leq \sigma \leq \frac{3}{4}, \\ \frac{2}{\sigma} & \text{for } \frac{3}{4} \leq \sigma \leq 1. \end{cases} \tag{1.2}$$

Combining the method of Karatsuba with Voronin's, Yashiro gave another proof for (1.2) in [19]. In 2007, Sankaranarayanan and Sengupta [16] investigated this problem when f is a Maass cusp form for $SL_2(\mathbb{Z})$, and proved the estimate

$$N_f(\sigma, T) \ll T^{\frac{4(1-\sigma)}{3-2\sigma}} \log^{26} T \quad \text{for} \quad \sigma \geq \frac{1}{2} + \frac{1}{\log T}.$$

Later, Xu [18] improved this result when $3/4 < \sigma < 1$ by showing that

$$N_f(\sigma, T) \ll T^{\frac{(8\sigma-5)(1-\sigma)}{-2\sigma^2+6\sigma-3}} \log^{57} T \quad \text{for} \quad \frac{3}{4} \leq \sigma < 1.$$

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