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A UNIQUE PAIR OF TRIANGLES

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Abstract. A rational triangle is a triangle with sides of rational lengths. In this short note, we prove that there exists a unique pair of a rational right triangle and a rational isosceles triangle which have the same perimeter and the same area. In the proof, we determine the set of rational points on a certain hyperelliptic curve by a standard but sophisticated argument which is based on the 2-descent on its Jacobian variety and Coleman's theory of p-adic abelian integrals.

1. Main theorem and its proof

A rational (resp. integral) triangle is a triangle with sides of rational (resp. integral) lengths. Such a triangle has arithmetic interest: For instance, every rational right triangle has the lengths of the sides $(k(1+x^2), k(1-x^2), 2kx)$ with positive rational numbers $k, x > 0$. We may check this fact from the Pythagorean theorem and the uniqueness of the prime decomposition, which are most elementary theorems in geometry and arithmetic respectively.

From the point of view of arithmetic, perimeter and area are fundamental invariants of a rational triangle. Therefore, it is natural to try to classify rational triangles by their perimeters and/or areas. Indeed, there are several works on construction of infinitely many pairs of rational triangles which have the same perimeters and areas (see e.g. [Bre06], [vL07] and references there).

A primitive triangle is an integral triangle such that the greatest common divisor of the lengths of its sides is 1. We can prove that there exists no pair of a primitive right triangle and a primitive isosceles triangle which have the same perimeter and the same area. We give an elementary proof of this fact in Appendix. How many such pairs are there in the category of rational triangles? In this short note, we give the complete answer to this question, that is, there exists only one such pair of triangles.

Theorem 1.1. *Up to similitude, there exists a unique pair of a rational right triangle and a rational isosceles triangle which have the same perimeter and the same area. The unique*

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