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A probabilistic generalization of the Stirling numbers of the second kind

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ABSTRACT

Associated to each random variable Y satisfying appropriate moment conditions, we introduce a different generalization of the Stirling numbers of the second kind. Some characterizations and specific examples of such generalized numbers are provided. As far as their applications are concerned, attention is focused in extending in various ways the classical formula for sums of powers on arithmetic progressions. Illustrations involving rising factorials, Bell polynomials, polylogarithms, and a certain class of Appell polynomials, in connection with appropriate random variables Y in each case, are discussed in detail.

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1. Introduction

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Unless otherwise specified, we assume throughout this paper that $f : \mathbb{R} \to \mathbb{R}$ is an arbitrary function, $m, n, N \in \mathbb{N}_0$, and $x \in \mathbb{R}$. Moreover, we set $z = it$, where i is the imaginary unit and $t \in \mathbb{R}$ with $|t| < r$, where $r > 0$ may change from line to line. We also denote by $I_n(x) = x^n$ the monomial function.

The celebrated formula for sums of powers on arithmetic progressions states that

$$\sum_{k=0}^{N} I_n(x+k) = \frac{B_{n+1}(x+N+1) - B_{n+1}(x)}{n+1}, \tag{1}$$

where $B_n(x)$ is the nth Bernoulli polynomial. Since the time of James Bernoulli (1655–1705), different generalizations of such sums have been obtained (see, for instance, Kannappan and Zhang [17], Guo and Zeng [13], Adell and Lekuona [3], Kim and Kim [18], and the references therein).

On the other hand, the usual mth forward difference of f is defined as

$$\Delta^m f(x) = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} f(x+k). \tag{2}$$

From our point of view, an interesting fact is that the sums in (1) can also be computed in terms of forward differences of the monomial function $I_n(x)$. Actually, we have (see, for instance, Rosen [26, p. 199] or Spivey [27])

$$\sum_{k=0}^{N} I_n(x+k) = \sum_{m=0}^{n \wedge N} \binom{N+1}{m+1} \Delta^m I_n(x), \tag{3}$$

where $n \wedge N = \min(n, N)$. Computationally, formulas (1) and (3) are equivalent in the sense that the computation of a sum of $N+1$ terms is reduced to the computation of a polynomial in N of degree $n+1$.

Finally, denote by

$$S(n, m; x) := \frac{\Delta^m I_n(x)}{m!}, \quad m \le n, \tag{4}$$

the Stirling polynomials of the second kind, so that

$$S(n, m) := S(n, m; 0), \quad m \le n, \tag{5}$$

are the classical Stirling numbers of the second kind (see Abramowitz and Stegun [1] or Roman [25, p. 60] for other equivalent definitions). Obviously, formula (3) can be rewritten in terms of $S(n, m; x)$.

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