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ON A STRONGER VERSION OF A QUESTION PROPOSED BY K. MAHLER

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ABSTRACT. In 1902, P. Stäckel proved the existence of a transcendental function $f(z)$, analytic in a neighborhood of the origin, and with the property that both $f(z)$ and its inverse function assume, in this neighborhood, algebraic values at all algebraic points. Based on this result, in 1976, K. Mahler raised the question of the existence of such functions which are analytic in $\mathbb{C}$. Recently, the authors answered positively this question. In this paper, we prove a much stronger version of this result by considering other subsets of $\mathbb{C}$.

1. INTRODUCTION

A *transcendental function* is a function $f(x)$ such that the only complex polynomial satisfying $P(x, f(x)) = 0$ for all x in the domain of f, is the null polynomial. For instance, the trigonometric functions, the exponential function, and their inverses are transcendental functions.

The study of the arithmetic behavior of transcendental functions at complex points has attracted the attention of many mathematicians for decades. The first result concerning this subject goes back to 1884, when Lindemann proved that the transcendental function e^z assumes transcendental values at all nonzero algebraic points. In 1886, Strauss tried to prove that an analytic transcendental function cannot assume rational values at all rational points in its domain. However, in 1886, Weierstrass supplied him with a counter-example and also stated that there are transcendental entire functions which assume algebraic values at all algebraic points. This assertion was proved in 1895 by Stäckel [8] who established a much more general result: for each countable subset $X \subseteq \mathbb{C}$ and each dense subset $Y \subseteq \mathbb{C}$, there exists a transcendental entire function f such that $f(X) \subseteq Y$. In another construction, Stäckel [9] produced a transcendental function $f(z)$, analytic in a neighborhood of the origin, and with the property that both $f(z)$ and its inverse function assume, in this neighborhood, algebraic values at all algebraic points. Based on this result, in 1976, Mahler [4, p. 53] suggested the following question

Question 1. *Does there exist a transcendental entire function*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

with rational coefficients a_n *and such that the image and the preimage of* $\overline{\mathbb{Q}}$ *under* f *are subsets of* $\overline{\mathbb{Q}}$*?*

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