Journal of Number Theory 194 (2019) 381–389

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

On a function introduced by Erdös and Nicolas

José Manuel Rodríguez Caballero

Département de Mathématiques, Université du Québec à Montréal, Case Postale 8888, Succ. Centre-ville, Montréal, Québec H3C 3P8, Canada

ARTICLE INFO

Article history:
Received 22 February 2018
Received in revised form 31 May 2018
Accepted 27 June 2018
Available online 17 July 2018
Communicated by L. Smajlovic

Keywords:
Distribution of divisors on short intervals
Generating function
Pythagorean triples

ABSTRACT

Text. Erdös and Nicolas [1] introduced an arithmetical function $F(n)$ related to divisors of n in short intervals $\left]\frac{t}{2}, t\right]$. The aim of this note is to prove that $F(n)$ is the largest coefficient of polynomial $P_n(q)$ introduced by Kassel and Reutenauer [2]. We deduce that $P_n(q)$ has a coefficient larger than 1 if and only if $2n$ is the perimeter of a Pythagorean triangle. We improve a result due to Vatne [7] concerning the coefficients of $P_n(q)$.

Video. For a video summary of this paper, please visit https://youtu.be/Ms9lrm4wLDQ.

1. Introduction

Erdös and Nicolas introduced in [1] the function

$$F(n) = \max\{\mathrm{q}_t(n) : \quad t \in \mathbb{R}_+^*\}, \tag{1}$$

where $\mathrm{q}_t(n) = \#\left\{d : \ d|n \text{ and } \frac{1}{2}t < d \leq t\right\}$, and they proved that

E-mail address: rodriguez_caballero.jose_manuel@uqam.ca.

https://doi.org/10.1016/j.jnt.2018.06.018

$$\lim_{x \to +\infty} \frac{1}{x} \sum_{n \leq x} F(n) = +\infty. \tag{2}$$

Kassel and Reutenauer introduced in [2] a q-analog of the sum of divisors, denoted $P_n(q)$, by means of the generating function

$$\prod_{m \geq 1} \frac{(1-t^m)^2}{(1-q\,t^m)\,(1-q^{-1}\,t^m)} = 1 + \left(q + q^{-1} - 2\right) \sum_{n=1}^{\infty} \frac{P_n(q)}{q^{n-1}}\, t^n \tag{3}$$

and they proved that, for $q = \exp\left(\frac{2\pi}{k}\sqrt{-1}\right)$, with $k \in \{2, 3, 4, 6\}$, this infinite product can be expressed by means of the Dedekind η-function (see [4]). A consequence of this coincidence is that the corresponding arithmetic functions $n \mapsto P_n(q)$, for each of the above-mentioned values of q, are related to the number of ways to express a given integer by means of a quadratic form (see [2] and [3]). The polynomials $P_n(q)$ are named *Kassel–Reutenauer q-analog of the sum of divisors*[5].

The aim of this paper is to prove the following theorem.

Theorem 1. *For each integer $n \geq 1$, the largest coefficient of $P_n(q)$ is $F(n)$.*

Using this result, we will derive that $P_n(q)$ has a coefficient larger than 1 if and only if $2\,n$ is the perimeter of a Pythagorean triangle. Also, we will prove that each nonnegative integer m is the coefficient of $P_n(q)$ for infinitely many positive integers n.

2. Proof of the main result

In order to simplify the notation in the proofs, we will consider two functions[1] $f : \mathbb{R} \longrightarrow \mathbb{R}_+^*$ and $g : \mathbb{R}_+^* \longrightarrow \mathbb{R}$, defined by

$$f(x) = \frac{1}{2}\left(x + \sqrt{8\,n + x^2}\right), \tag{4}$$

$$g(y) = y - \frac{2\,n}{y}. \tag{5}$$

Lemma 2. *The functions $f(x)$ and $g(y)$ are well-defined, strictly increasing and mutually inverse. Furthermore, $g(y)$ satisfies the identity*

$$g\,(y) = -g\left(\frac{2\,n}{y}\right). \tag{6}$$

Proof. It follows in a straightforward way from the explicit expressions (4) and (5) that $f(x)$ and $g(y)$ are well-defined and strictly increasing. In particular, the inequality $|x| < \sqrt{8\,n + x^2}$ guarantees that $f(x) \in \mathbb{R}_+^*$ for all $x \in \mathbb{R}$.

[1] The function $g(y)$ was implicitly used in Proposition 2.2, in [4].