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The unreasonable rigidity of Ulam sequences

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A B S T R A C T

An Ulam sequence $U(a, b)$ is defined as a sequence starting with integers a, b such that $0 < a < b$, and such that every subsequent term is the smallest integer that can be written as the sum of distinct previous terms in exactly one way. We investigate a new rigidity phenomenon for families $U(a, b)$, where a is fixed and we allow b to increase, centering around a result proved using a novel model theoretic argument. For the specific case $U(1, n)$, we provide an algorithm for computing relevant coefficients, together with a proof of correctness.

1. Introduction and main results

1.1. Introduction

In 1964, Ulam introduced his eponymous sequence

$$1, 2, 3, 4, 6, 8, 11, 13, 16, 18, 26, 28, 36, 38, 47, 48, 53, 57, 62, 69, 72, 77, 82, 87, 97 \dots$$

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defined recursively so that the first two terms are $1, 2$ and each subsequent term is the smallest integer that can be written as the sum of two distinct prior terms in a unique way [12]—we shall denote this sequence as $U(1, 2)$, for reasons that will become evident later. Ulam was interested in determining the growth of this sequence—the best known bound is that the Ulam sequence grows no faster than the Fibonacci sequence. However, the Fibonacci sequence grows exponentially, whereas experimental data suggests that the Ulam sequence has positive density about 0.079 [6].

Interest in the Ulam sequence has seen a resurgence in recent years—see [5,7,8,11]. However, despite a multitude of new phenomena being discovered, even some very basic question remain unanswered. It seems the great difficulty in proving anything about Ulam sequences is that we understand so little about their structure—they almost seem random, and have been described in the literature as "erratic" [10]. Our present goal is to show that, in contrast, families of Ulam sequences varying in some parameter can be startlingly rigid.

1.2. Summary of main results

Let $U(a, b)$ denote the generalized Ulam sequences starting with integers a, b, such that each subsequent term is the smallest integer that can be written as the sum of two distinct preceding terms in exactly one way. An important class of examples is the family $U(1, n)$ where $n \in \mathbb{Z}_{>1}$. The first few terms of $U(1, 2)$, $U(1, 3)$, $U(1, 4)$, $U(1, 5)$, and $U(1, 6)$ are given below.

$$
\begin{array}{lcl}
U(1,2) &=& 1,\ 2,\ 3,\ 4,\ 6,\ 8,\ \ 11,\ 13,\ 16,\ 18,\ 26,\ 28,\ 36,\ 38,\ 47,\ 48\ldots \\
U(1,3) &=& 1,\ 3,\ 4,\ 5,\ 6,\ 8,\ \ 10,\ 12,\ 17,\ 21,\ 23,\ 28,\ 32,\ 34,\ 39,\ 43\ldots \\
U(1,4) &=& 1,\ 4,\ 5,\ 6,\ 7,\ 8,\ \ 10,\ 16,\ 18,\ 19,\ 21,\ 31,\ 32,\ 33,\ 42,\ 46\ldots \\
U(1,5) &=& 1,\ 5,\ 6,\ 7,\ 8,\ 9,\ \ 10,\ 12,\ 20,\ 22,\ 23,\ 24,\ 26,\ 38,\ 39,\ 40\ldots \\
U(1,6) &=& 1,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12,\ 14,\ 24,\ 26,\ 27,\ 28,\ 29,\ 31,\ 45\ldots
\end{array}
$$

Startlingly, there appears to be a simple formula in n for the first few terms of each of these sequences—specifically,

$$U(1, n) \cap [1, 3n] = \{1\} \cup \{n, n + 1, \ldots, 2n\} \cup \{2n + 2\}.$$

Furthermore, for $n \geq 4$, this pattern seems to extend further.

$$
\begin{aligned}
U(1, n) \cap [1, 6n] = \{1\} &\cup \{n, n + 1, \ldots, 2n\} \\
&\cup \{2n + 2\} \cup \{4n\} \\
&\cup \{4n + 2, 4n + 3, \ldots 5n - 1\} \\
&\cup \{5n + 1\}.
\end{aligned}
$$

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