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A Dirichlet's principle for the k-Hessian

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The k-Hessian operator σ_k is the k-th elementary symmetric function of the eigenvalues of the Hessian. It is known that the k-Hessian equation $\sigma_k(D^2u) = f$ with Dirichlet boundary condition $u = 0$ is variational; indeed, this problem can be studied by means of the k-Hessian energy $-\int u\sigma_k(D^2u)$. We construct a natural boundary functional which, when added to the k-Hessian energy, yields as its critical points solutions of k-Hessian equations with general non-vanishing boundary data. As a consequence, we establish a Dirichlet's principle for k-admissible functions with prescribed Dirichlet boundary data.

1. Introduction

Let $X \subset \mathbb{R}^n$ be a bounded smooth domain with boundary $M = \partial X$. The usual trace inequality states that

$$-\int_X u\Delta u \, dx + \oint_M f u_n d\mu \geq \oint_M f(u_f)_n d\mu \tag{1.1}$$

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for all $f \in C^\infty(M)$ and all $u \in C^\infty(\overline{X})$ such that $u|_M = f$, where u_n denotes the derivative of u with respect to the outward-pointing normal along M, u_f is the harmonic function in X such that $u_f|_M = f$, and dx, $d\mu$ are the volume forms on X and M, respectively. Integrating (1.1) by parts gives the Dirichlet's principle for harmonic functions:

$$\int_X |\nabla u|^2 \, dx \geq \int_X |\nabla u_f|^2 \, dx$$

for all u with $u|_M = f$. A standard density argument implies that the trace $\mathrm{tr} u := u|_M$ extends to a bounded linear operator $\mathrm{tr}\colon W^{1,2}(\overline{X}) \to W^{1/2,2}(M)$ and the extension $E(f) := u_f$ extends to a bounded linear operator $E\colon W^{1/2,2}(M) \to W^{1,2}(\overline{X})$ such that $\mathrm{tr} \circ E$ is the identity.

The trace inequality (1.1) is a useful tool in many analytic and geometric problems. For example, the Dirichlet-to-Neumann map $f \mapsto (u_f)_n$ is a pseudodifferential operator with principle symbol $(-\Delta)^{1/2}$; indeed, it is the operator $(-\Delta)^{1/2}$ when $X = \mathbb{R}^n_+$ is the upper half-plane. Thus (1.1) relates the energy of the local operator $-\Delta$ to the energy of the nonlocal Dirichlet-to-Neumann operator, providing a useful tool for establishing estimates for PDEs stated in terms of the latter operator. This strategy provides a key motivation for the approach of Caffarelli and Silvestre [2] for studying fractional powers of the Laplacian.

The purpose of this article is to establish an analogue of (1.1) in terms of the k-Hessian energy $\sigma_k(D^2u)$. Here D^2u denotes the Hessian of u and $\sigma_k(A)$ denotes the k-th elementary symmetric function

$$\sigma_k(A) := \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k},$$

of the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A. The Dirichlet problem

$$\begin{cases} \sigma_k(D^2u) = F(x,u), & \text{in } X, \\ u = f(x), & \text{on } M \end{cases} \tag{1.2}$$

has been well-studied for functions u in the elliptic k-cone

$$\Gamma_k^+ := \left\{ u \in C^\infty(\overline{X}) \;\middle|\; \sigma_j(D^2u) > 0, 1 \leq j \leq k \right\}; \tag{1.3}$$

e.g. [1,5,7,9,10]. Note that the existence of a solution to (1.2) requires that M be $(k-1)$-convex [1]; i.e. the second fundamental form L of M must satisfy $\sigma_j(L) > 0$ for $1 \leq j \leq k-1$. Of particular relevance to us is that when $F = 0$, the existence and uniqueness of a solution $u \in \overline{\Gamma_k^+}$ of the degenerate Dirichlet problem (1.2) is known [5,8], where $\overline{\Gamma_k^+}$ is the $C^{1,1}(\overline{X})$-closure of the elliptic k-cone (1.3). Our contribution is the following Dirichlet's principle for such solutions: