Journal of Functional Analysis 275 (2018) 2978–3064

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

An explicit trace formula of Jacquet–Zagier type for Hilbert modular forms

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ARTICLE INFO

Article history:
Received 11 October 2017
Accepted 13 September 2018
Available online 21 September 2018
Communicated by P. Delorme

MSC:
primary 11F72
secondary 11F67

Keywords:
Trace formulas
Adjoint L-functions

ABSTRACT

We give an exact formula of the average of adjoint L-functions of holomorphic Hilbert cusp forms with a fixed weight and a square-free level, which is a generalization of Zagier's formula known for the case of elliptic cusp forms on $\mathrm{SL}_2(\mathbb{Z})$. As an application, we prove that the Satake parameters of Hilbert cusp forms with a fixed weight and with growing square-free levels are equidistributed in an ensemble constructed by values of the adjoint L-functions.

1. Introduction

1.1. Background and motivation

In [33], Zagier proposed an elegant way to compute the traces of Hecke operators on the space of elliptic cusp forms by means of the Rankin–Selberg method. This is a more

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https://doi.org/10.1016/j.jfa.2018.09.009
0022-1236/

direct way than the Selberg's one because it does not need a deliberate rearrangement for renormalization of divergent terms which are inevitably produced by truncation process. Later, a similar study was conducted for Maass forms on $\mathrm{SL}_2(\mathbb{Z})$ in [34], and for general cusp forms on the adelization $\mathrm{GL}_2(\mathbb{A}_F)$ with an arbitrary number field F in the work of Jacquet and Zagier [10]. The formula proved in [10] can be viewed as a deformation by a complex parameter z of the usual Arthur–Selberg trace formula for GL(2) because the latter one is expected to be recovered from the former one as the residue at $z = 1$. Although their general formula is less explicit than Zagier's one, it provides us with a different proof of holomorphicity of the symmetric square L-function of a cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_F)$, which was first proved by Shimura [20] in a classical setting and later was generalized to an adelic setting by [5]. The paper [33] also gave an application of the formula to the proof of the algebraicity of the critical values of the symmetric square L-functions for elliptic cusp forms, which was independently obtained by Strum [21] based on [20]. Mizumoto [16] and Takase [27] extended Zagier's method to Hilbert cusp forms under the assumption that the narrow class number of the base field is one. For application to special values, the explicit nature of their formula is crucial.

In this paper, motivated by these works and intending further potential applications, we shall calculate Jacquet–Zagier's trace formula for holomorphic Hilbert cusp forms as explicitly as possible when the level is square-free without the assumption on the class number of the base field F. For a technical reason, we assume that the prime 2 splits completely in F. We remark that this assumption is mild enough to include the interesting cases $F = \mathbb{Q}$ and $F = \mathbb{Q}(\sqrt{D})$ with $D > 0$ and $D \equiv 1 (\mathrm{mod}\ 8)$, and our formulas for Hilbert modular forms with large levels are new even for $F = \mathbb{Q}$. Since we use the matrix coefficients of discrete series representations at archimedean places, which are not compactly supported contrary to the test functions dealt in [10], we have to modify the convergence proof of the geometric side in a substantial way. Moreover, we completely calculate all local terms in the formula for a large class of test functions. As an application, we prove an equidistribution result of Satake parameters in the ensemble defined by the symmetric square L-functions $L(z, \pi; \mathrm{Ad})$ of holomorphic Hilbert cusp forms π of a fixed weight with the varying square-free levels, in such a way that a part of the famous Serre's equidistribution theorem ([17, Théorème 1]) is recovered from our formula by the specialization at $z = 1$. The non-vanishing of the symmetric square L-function of an elliptic modular form at a point in the critical strip has been pursued by many authors ([11], [13], [2]). As a corollary to our asymptotic formula, we obtain infinitely many Hilbert cusp forms with a fixed weight of growing levels whose symmetric square L-functions are non-vanishing at a given point in the critical strip. Our method is not the Rankin–Selberg method in the accurate sense, because the Eisenstein series is not unfolded on the convergence region as was done in [33] and [10]; actually, the same proof works if the Eisenstein series is replaced with a Maass cusp form. To illustrate the robustness of our method, we deduce Theorem 10.1 which is viewed as an adelic version of Gon's formula in an opposite setting to [6].