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Symmetrically separated sequences in the unit sphere of a Banach space ☆

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We prove the symmetric version of Kottman's theorem, that is to say, we demonstrate that the unit sphere of an infinite-dimensional Banach space contains an infinite subset A with the property that $\|x \pm y\| > 1$ for distinct elements $x, y \in A$, thereby answering a question of J.M.F. Castillo. In the case where X contains an infinite-dimensional separable dual space or an unconditional basic sequence, the set A may be chosen in a way that $\|x \pm y\| \geqslant 1 + \varepsilon$ for some $\varepsilon > 0$ and distinct $x, y \in A$. Under additional structural properties of X, such as non-trivial cotype, we obtain quantitative estimates for the said ε. Certain renorming results are also presented.

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Kottman theorem
Symmetric Kottman constant

1. Introduction

Kottman's theorem [29], asserting that the unit sphere of an infinite-dimensional normed space contains a sequence of points whose mutual distances are strictly greater than one, sparked a new insight on the non-compactness of the unit ball in infinite dimensions. Elton and Odell [17] employed methods of infinite Ramsey theory to improve Kottman's theorem significantly by showing that the unit sphere of an infinite-dimensional normed space contains a sequence $(x_n)_{n=1}^{\infty}$ such that $\|x_n - x_k\| \geqslant 1 + \varepsilon$ $(k, n \in \mathbb{N}, k \neq n)$ for some $\varepsilon > 0$. Even though the proof of Kottman's theorem has been greatly simplified over time ([14, pp. 7–8]), it was only recently when a new (still Ramsey-theoretic though) proof of the Elton–Odell theorem was obtained ([20]).

It is perhaps no surprise that the ε appearing in the statement of the Elton–Odell theorem is intimately related to the geometry of the underlying space. Indeed, in the case of the space ℓ_p $(1 \leqslant p < \infty)$ it cannot be greater than the attained bound $2^{1/p} - 1$ (see, *e.g.*, [3, p. 31]). Thus, studying geometric or structural properties of the space will often help in identifying possible lower bounds for separation constants of sequences in the unit sphere of the space. For example, Kryczka and Prus proved the quite remarkable result saying that in the unit sphere of a non-reflexive Banach space one may find a $\sqrt[5]{4}$-separated sequence ([30]). Further quantitative estimates of the said lower bound expressed in terms of various moduli of convexity and related results may be found in [11,13,34,37,40].

The main objective of the paper is to revisit and investigate the above-mentioned results in the setting of symmetric separation: let us say that a subset A of a normed space is *symmetrically* $(\delta+)$*-separated* (respectively, *symmetrically* δ*-separated*) when $\|x \pm y\| > \delta$ (respectively, $\|x \pm y\| \geqslant \delta$) for any distinct elements $x, y \in A$ $(\delta > 0)$. J.M.F. Castillo and P.L. Papini asked whether there is a symmetric version of the Elton–Odell theorem ([10, Problem 1]), however according to Castillo ([9]) prior to this research, it has not been known whether the unit sphere of an infinite-dimensional Banach space contains a symmetrically $(1+)$-separated sequence.

Castillo and Papini proved that the answer is affirmative for uniformly non-square spaces and for $\mathscr{L}_\infty$-spaces (consult [10] for more details). Also, although not stated explicitly, it follows from the proof of the main result of [13] that the answer is affirmative for asymptotically uniformly convex spaces in which case the lower bound for the symmetric separation constant is expressed in terms of the so-called modulus of asymptotic uniform convexity. Certainly the unit spheres of both ℓ_1 and c_0 contain symmetrically 2-separated sequences (in the former case plainly the standard vector basis is an example of such sequence, in the latter case one may take $x_n = -e_{n+1} + \sum_{k=1}^{n} e_k$ $(n \in \mathbb{N})$, where $(e_n)_{n=1}^{\infty}$ denotes the unit vector basis of c_0). Consequently, if X contains an iso-

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