PSEUDO-FINSLER SPACES MODELED ON A PSEUDO-MINKOWSKI SPACE

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We adopt a vierbein formalism to study pseudo-Finsler spaces modeled on a pseudo-Minkowski space. We show that it is possible to obtain closed expressions for most of the geometric objects of the theory, including Berwald's curvature, Landsberg's tensor, Douglas' curvature, nonlinear connection and Ricci scalar. These expressions are particularly convenient in computations since they factor the dependence on the base and the fiber. As an illustration, we study Lorentz-Finsler spaces modeled on the Bogoslovsky Lorentz-Minkowski space, and give sufficient conditions which guarantee the Berwald property. We then specialize to a recently proposed Finslerian pp-wave metric. Finally, the paper points out that nontrivial Berwald spaces have necessarily indicatrices which admit some nontrivial linear group of symmetries.

Keywords: Finsler geometry, Minkowski space, Berwald space.

1. Introduction

In this work by pseudo-Minkowski space we mean a pair (V, L) where V is an m-dimensional vector space and $L\colon \bar{\omega} \to \mathbb{R}$ is a function defined on the closure of an open cone[2] $\omega \subset V$ with vertex in the origin, such that

(a) $L \in C^4(\omega) \cap C^0(\bar{\omega})$,
(b) $\forall s > 0$, and $y \in V$ we have $L(sy) = s^2 L(y)$,
(c) the Hessian $g := (\partial_\mu \partial_\nu L) \mathrm{d}y^\mu \otimes \mathrm{d}y^\nu$ is nondegenerate on ω.

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[2]We do not assume it to be convex.

It is understood that the Hessian is calculated using any set of linear coordinates on V (namely a dual basis on V^*). More specific theories are possible, for instance, if g is positive definite and $\omega = V\backslash 0$ one speaks of Minkowski space, while if g has Lorentzian signature and ω is a sharp convex cone one speaks of Lorentz–Minkowski space. We warn the reader that already in the positive definite case the concept of Minkowski space is not homogeneously defined in the literature (compare [1, 2]).

If we were concerned with just Minkowski and Lorentz–Minkowski spaces we would probably add in the definition the condition

(d) $L|_\omega \neq 0$ and $L|_{\partial\omega} = 0$.

However, we shall not use it in our derivations (so vector spaces endowed with a nondegenerate bilinear form are included in the definition).

A pseudo-Finsler space is instead a pair $(M, \mathscr{L})$ where M is an m-dimensional manifold and $\mathscr{L} : \bar{\Omega} \to \mathbb{R}$, $\Omega \subset TM$, $\pi(\Omega) = M$, with the property that if we define $\Omega_x := \pi^{-1}(x) \cap \Omega$, $\mathscr{L}_x := \mathscr{L}|_{\bar{\Omega}_x}$, $\Omega_x \subset T_xM$, then the pair $(T_xM, \mathscr{L}_x)$ is a pseudo-Minkowski space with a signature independent of x. One can assume various differentiability conditions on the dependence of $\mathscr{L}$ on x, for our calculations C^2 will suffice. Of course, one could define Finsler and Lorentz–Finsler spaces adding the conditions introduced above for pseudo-Minkowski spaces.

REMARK 1. The domain of L is really of little importance for the purpose of this work since all our calculations are local over TM. We have chosen a closed cone domain because the causality aspects of these theories can be studied with the help of the theory of differential inclusions.

This paper is concerned with a special type of pseudo-Finsler space, namely those for which all pseudo-Minkowskian tangent spaces $(T_xM, \mathscr{L}_x)$ are modeled on the same pseudo-Minkowski space (V, L). This means that locally we can find a linear isomorphism (sufficiently differentiable in $x \in U \subset M$)

$$\varphi_x : T_xM \to V, \tag{1}$$

such that $\varphi_x(\Omega_x) = \omega$, and $L(\varphi_x(y)) = \mathscr{L}_x(y)$ for every $y \in \bar{\Omega}_x$. Parallelizable manifolds admit structures of this type and parallelizability can be relaxed provided (V, L) admits a Lie group G of linear isomorphisms (they preserve $\mathscr{L}$ and its domain). In that case it is sufficient that M admits a G-structure [3]. Our considerations will be of local character so we shall not enter into a more detailed discussion [3]. The idea of Finsler space modeled on the same Minkowski space has been first introduced and investigated by Ichijyo [3] and further progress has been obtained by Aikou in [4]. Matsumoto [5], Izumi [6], Sakaguchi [7], and Asanov and Kirnasov [8] had also considered similar spaces referring to them as *1-form Finsler spaces*. Szilasi and Tamássy call them *affine deformations of Minkowski spaces* [9], while Libing Huang, and Bartelmeß and Matveev borrow from Zhongmin Shen the terminology *single colored* or *monochromatic* Finsler spaces [10, 11].

Roughly speaking, while in pseudo-Finsler spaces the anisotropic features change from point to point and can even be absent in some open subset of M, in pseudo-