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GENERALIZED INTEGRABLE HIERARCHIES OF AKNS TYPE, SUPER DIRAC TYPE AND SUPER NLS–MKDV TYPE

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Integrable hierarchies provide many important physical models. Firstly, with the help of symbolic computation software Maple, two generalized integrable hierarchies of Ablowitz–Kaup–Newell–Segur (AKNS) type are constructed from the matrix spectral problem associated with the Lie algebra $\mathfrak{sl}(2)$. Then, two generalized integrable hierarchies of super Dirac type and super nonlinear Schrödinger-modified Korteweg–de Vries (NLS–mKdV) type are obtained to illustrate the use of the super Lie algebra $\mathfrak{B}(0, 1)$.

Keywords: integrable hierarchy, matrix spectral problem, Lie algebra $\mathfrak{sl}(2)$, super Lie algebra $\mathfrak{B}(0, 1)$.

1. Introduction

During the last forty years, integrable system theory has achieved great success, and its development is still quite exciting. Matrix spectral problems associated with given Lie algebras such as $\mathfrak{sl}(2)$, $\mathfrak{so}(4)$ and $\mathfrak{sl}(m+1)$ are crucial keys to construct integrable hierarchies. A series of systematic methods has been developed to study integrable hierarchies such as the Tu scheme, the nonlinearization technique, the inverse scattering transformation, the Bäcklund transformation, the Darboux transformation and the Hirota's bilinear transformation. There has been a lot of interesting examples including the Ablowitz–Kaup–Newell–Segur (AKNS), the Kaup–Newell (KN), the Wadati–Konno–Ichikawa (WKI), the Korteweg–de Vries (KdV), the modified KdV, the Benjamin–Ono, the Dirac, the coupled Harry–Dym and the coupled Burgers integrable hierarchies [1–3, 5–7, 12, 13, 15–18, 25–27].

In order to construct an integrable hierarchy with two physical quantities, the three-dimensional real special linear Lie algebra $\mathfrak{sl}(2)$ is a very important tool. It is well known that this algebra, consisting of trace-free 2×2 matrices, is simple and has the basis

$$\gamma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \gamma_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \gamma_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

whose commutator relations are $[\gamma_1, \gamma_2] = 2\gamma_2$, $[\gamma_1, \gamma_3] = -2\gamma_3$ and $[\gamma_2, \gamma_3] = \gamma_1$. Thus, we can give a brief description of the procedure for building an integrable hierarchy associated with $\mathfrak{sl}(2)$, which is called the Tu scheme [15–18, 23, 49].

Step 1: We need to construct an appropriate spectral matrix U to form a spatial matrix spectral problem $\phi_x = U\phi$.

Step 2: In order to obtain the recursion relation, we need to solve the stationary zero-curvature equation $W_x = [U, W]$.

Step 3: We need to construct temporal matrix spectral problems $\phi_{t_m} = V^{[m]}\phi$ so that the zero-curvature equations $U_{t_m} - V_x^{[m]} + \left[U, V^{[m]}\right] = 0$ can generate an integrable hierarchy.

Recently, there has been great interest in generalising integrable systems to their super integrable analogues. Meanwhile, mathematical physics has developed a new fruitful conception of super model theories, where the anticommuting (odd) variables are equally treated with the usual commuting (even) variables. For example, three kinds of super extensions for the famous KdV equation have been obtained by the authors in [8, 9, 37]. These super KdV equations are all derived from some super integrable hierarchies, therefore, it is important to construct new super integrable hierarchies. In fact, many super integrable hierarchies have also been constructed, such as the super AKNS, the super Dirac, the super KdV, the super nonlinear Schrödinger–modified Korteweg–de Vries (NLS–mKdV) and the super Kadomtsev–Petviashvili (KP) integrable hierarchies [4, 8, 9, 11, 14, 15, 21–24, 28–30, 32–46, 48, 49, 51, 52].

In general, a super integrable hierarchy can be obtained by the super Lie algebra $\mathfrak{B}(0, 1)$, which is a subalgebra of the super Lie algebra $\mathfrak{sl}(m/n)$. Here we take the basis $\{e_1, e_2, e_3, e_4, e_5\}$ of $\mathfrak{B}(0, 1)$ to be [23, 49]

$$e_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad e_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad e_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$e_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \qquad e_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$