ON THE SHARPNESS OF SPECTRAL ESTIMATES FOR GRAPH LAPLACIANS

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We prove that the upper spectral estimate for quantum graphs due to Berkolaiko–Kennedy–Kurasov–Mugnolo [5] is sharp.

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1. Brief introduction

Quantum graphs is a rapidly developing branch of modern mathematical physics with important application in neighbouring areas including discrete mathematics [4, 7, 10, 14]. Spectral estimates, in particular estimates involving the lowest eigenvalues or the spectral gap, form an important research direction [1–3, 5, 6, 8, 11–13, 15].

The goal of the current note is to show that one of the upper estimates proven recently [5] is in fact sharp.

2. Notation

A finite compact metric graph $\Gamma = (E, V)$ is composed of a set of edges $E = \{[x_{2i-1}, x_{2i}]\}_{i=1}^N$ and a set of vertices V that is a partition of the set of the endpoints of the edges. The Laplace operator $-d^2/dx^2$ on Γ acts on the functions defined on the edges. To make the operator self-adjoint we introduce **standard vertex conditions**[1] at the vertices $v \in V$,

$$\begin{cases} u \text{ is continuous in } v, \\ \sum_{x_j \in v} \partial u(x_j) = 0, \end{cases} \tag{1}$$

where $\partial u(x_j)$ denotes the normal derivative of u in the direction inside the edge. In particular, if $\deg(v) = 1$, then the standard condition reduces to the usual Neumann condition $u' = 0$. Summing up, the standard Laplacian $L(\Gamma)$ is defined in the Hilbert

[1] These conditions are sometimes called Neumann–Kirchhoff or natural.

space $L_2(\Gamma) = \oplus_{n=1}^N L_2[x_{2n-1}, x_{2n}]$ on the domain of functions from the Sobolev space $\bigoplus_{n=1}^N \mathcal{W}_2^2[x_{2n-1}, x_{2n}]$ satisfying in addition the standard vertex conditions (1).

Sometimes we are going to consider Dirichlet conditions at certain vertices of degree one $u(v) = 0$. The corresponding operator will be called the Dirichlet Laplacian and denoted by L^D. We hope that it will be clear at which vertices the Dirichlet conditions are assumed.

Both the standard and the Dirichlet Laplacians are self-adjoint operators in the Hilbert space $L_2(\Gamma)$. Their spectra are purely discrete with unique accumulation point ∞, provided the number of edges is finite [7, 10, 14]. Assume the graph Γ is connected, then the first eigenvalue (*i.e.* the lowest one) has multiplicity one, therefore we are going to enumerate the eigenvalues in the increasing order,

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots . \tag{2}$$

For the standard Laplacian the ground state is given by the constant function with the corresponding eigenvalue $\lambda_1 = 0$.

3. Eigenvalue estimate due to Berkolaiko–Kennedy–Kurasov–Mugnolo [5]

PROPOSITION 1 (Theorem 4.9 from [5]). *Let Γ be a compact finite metric graph with $|D|$ Dirichlet and $|N|$ Neumann degree-one vertices and β independent cycles. Then the following eigenvalue estimate holds for all $\lambda_n \in \sigma(\Gamma)$*

$$\lambda_n(\Gamma) \leq \left(\frac{\pi}{\mathcal{L}}\right)^2 \left(n - 2 + |D| + \frac{|N|}{2} + \frac{3\beta}{2}\right)^2 . \tag{3}$$

Here $\mathcal{L}$ is the total length of the graph, $\mathcal{L} = \sum_{n=1}^N (x_{2n} - x_{2n-1})$. The standard vertex conditions are assumed at all the vertices of degree ≥ 2. The graph is supposed to be connected.

The estimate was first proven for trees ($\beta = 0$) and it was realised that Dirichlet and Neumann boundary vertices contribute with factors 1 and 1/2, respectively. The estimate was generalised for $\beta \neq 0$ in [5] by cutting each cycle in the graph and introducing a new Neumann vertex. Cutting the cycle is a rank-one perturbation, which may shift the eigenvalues by one step, while the new Neumann boundary vertex gives another contribution of 1/2, hence a factor $1 + 1/2 = 3/2$ appears in front of β.

The authors mention that in their opinion the obtained estimate in not sharp (see [5]);

> ... *informal scaling arguments suggest that, for large β, the factor in front of β ... should be* 1 *instead of the current* $\frac{3}{2}$.

The aim of this work is to provide explicit examples proving that the estimate is on the contrary sharp. Searching for an example showing that the estimate is sharp we tried to find a graph with a high multiplicity of eigenvalues and easy to analyse analytically (more motivation can be found in Section 5). Windmill graph provides one such example.