MULTIPARTITE QUANTUM CORRELATIONS: SYMPLECTIC AND ALGEBRAIC GEOMETRY APPROACH

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We review a geometric approach to classification and examination of quantum correlations in composite systems. Since quantum information tasks are usually achieved by manipulating spin and alike systems or, in general, systems with a finite number of energy levels, classification problems are usually treated in frames of linear algebra. We proposed to shift the attention to a geometric description. Treating consistently quantum states as points of a projective space rather than as vectors in a Hilbert space we were able to apply powerful methods of differential, symplectic and algebraic geometry to attack the problem of equivalence of states with respect to the strength of correlations, or, in other words, to classify them from this point of view. Such classifications are interpreted as an identification of states with 'the same correlations properties', i.e. ones that can be used for the same information purposes, or, from yet another point of view, states that can be mutually transformed one to another by specific, experimentally accessible operations. It is clear that the latter characterization answers the fundamental question 'what can be transformed into what *via* available means?'. Exactly such an interpretation, i.e. in terms of mutual transformability, can be clearly formulated in terms of actions of specific groups on the space of states and is the starting point for the proposed methods.

Keywords: quantum entanglement, reduced density matrices, momentum map, entanglement polytopes, symplectic reduction, invariant polynomials, null cone.

1. Introduction

Quantum entanglement — a direct consequence of linearity of quantum mechanics and the superposition principle — is one of the most intriguing phenomena distinguishing the quantum and classical description of physical systems. Quantum correlated (e.g. entangled) states of composite systems possess features unknown in the classical world, like the seemingly paradoxical nonlocal properties exhibited by the famous Einstein–Podolsky–Rosen analysis of completeness of the quantum theory. Recently, with the development of quantum information theory they came to prominence as the main resource for several applications aiming at speeding up and making more secure information transfers (see, e.g. [1]). A novel kind of quantum correlations, called *quantum discord*, different from entanglement, but also absent in the macroscopic world, was discovered [2, 3] adding one more element to "the mysteries of quantum mechanics" as seen from the classical point of view.

Although typically a quantum system, such as, for instance, a harmonic oscillator or a hydrogen atom, is described in terms of an infinite-dimensional Hilbert space, for most quantum-information applications the restriction to finite dimensions suffices, since usually the active role in information processing is played only by spin degrees of freedom or only few energy levels are excited during the evolution.

From the mathematical point of view such finite-dimensional quantum mechanics seems to mount a smaller challenge than in the infinite-dimensional case—the tool of choice here is linear algebra rather than functional analysis. Nevertheless, the understanding of correlations in multipartite finite-dimensional quantum systems is still incomplete, both for systems of distinguishable particles [4] as well as for the ones consisting of nondistinguishable particles like bosons and fermions [5–8].

The statistical interpretation of quantum mechanics disturbs a bit the simple linear-algebraic approach to quantum mechanics—vectors corresponding to a state (elements of a finite-dimensional Hilbert space $\mathcal{H}$) should be of unit norm. Obviously, physicists are accustomed to cope with this problem in a natural way by "normalizing the vector and neglecting the global phase". Nevertheless, it is often convenient to implement this prescription by adopting a suitable mathematical structure, the projective space $\mathbb{P}(\mathcal{H})$, already from the start[1]. The projective space is obtained from the original Hilbert space $\mathcal{H}$ by identifying vectors[2] differing by a scalar, complex, nonzero factor, $|\psi\rangle \cong c|\psi\rangle$. We will denote elements (points) of $\mathbb{P}(\mathcal{H})$ by u, v, x, etc. and, if we want to identify a particular equivalence class of the vector $|\psi\rangle$, by $[\psi]$, etc.

Obviously, both approaches, the linear-algebraic (plus normalization and neglecting the global phase) picture and the projective one are equivalent. Following the former, we loose linearity, so which advantages we could expect instead? We answered this question in our paper [9], where we propose, by working in the projective space, to apply completely new (in this context) techniques to analyze the phenomenon of entanglement. The approach has given a deeper insight into the unexpectedly rich geometric structure of the space of states and enabled the use of recently developed advanced methods of complex differential, algebraic and symplectic geometry.

The most efficient characterization of quantum correlations is achieved by identifying states that are 'equally correlated' or, in other words, states that can be

[1] Equivalently, it is possible to incorporate the redundancy of the global phase by identifying pure states with orthogonal projectors onto one-dimensional subspaces of $\mathcal{H}$. However, for the sake of convenience, in this exposition we decided to treat pure states as elements of $\mathbb{P}(\mathcal{H})$.

[2] We will use exchangeably the Dirac notation, $|\psi\rangle$, etc., and the short one ψ etc. for elements (vectors) of $\mathcal{H}$.