A NOTE ON INCOMPRESSIBILITY OF RELATIVISTIC FLUIDS AND THE INSTANTANEITY OF THEIR PRESSURES

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We introduce a natural notion of incompressibility for fluids governed by the relativistic Euler equations on a fixed background spacetime, and show that the resulting equations reduce to the incompressible Euler equations in the classical limit as $c \to \infty$. As our main result, we prove that the fluid pressure of solutions of these incompressible "relativistic" Euler equations satisfies an elliptic equation on each of the surfaces orthogonal to the fluid four-velocity, which indicates infinite speed of propagation.

Keywords: relativity, incompressible Euler equations, elliptic PDE's.

1. Introduction

It is intriguing to study problems of classical fluid mechanics on a relativistic level, since the character of the underlying equations may change [5]. However, as shown in this note, care must be taken when addressing incompressible fluids, since infinite speeds of propagation should occur when requiring the particle number density to be constant, thus violating the axioms of relativity. This paper gives mathematical context to the expectation that incompressibility may in general not be a meaningful concept for relativistic fluids.

The study of incompressible fluids in general relativity began with Karl Schwarzschild's work [12], addressing the special case of a static fluid. A general model analogous to incompressible fluids has been introduced by Lichnerowicz [6] (see also [4]), by requiring that wave speed equals the speed of light. However, this model does not capture the rigid motion of nonrelativistic incompressible fluids in the relativistic setting.

Our starting point here is to assume such rigid motion by requiring the particle number density N to be constant along the fluid flow,

$$u^\sigma \partial_\sigma N = 0, \tag{1.1}$$

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where u denotes the fluid 4-velocity and ∂_σ partial differentiation in some fixed coordinate system, and we always sum over repeated upper and lower indices. The relativistic Euler equations are

$$\nabla_\mu(Nu^\mu) = 0, \tag{1.2}$$

$$\nabla_\nu T^{\mu\nu} = 0, \tag{1.3}$$

where T is the energy-momentum tensor of a perfect fluid,

$$T^{\mu\nu} \equiv (\rho + p)u^\mu u^\nu + pg^{\mu\nu}, \tag{1.4}$$

for ρ the energy density and p the pressure of the fluid. We assume a fixed background spacetime $\mathcal{M}$ with metric tensor g of signature $(-+++)$ and we raise and lower indices by contraction with g. We work in natural units ($c = 1$) and assume the fluid 4-velocity u to be normalized to

$$u^\sigma u_\sigma \equiv g_{\mu\nu}u^\mu u^\nu = -1. \tag{1.5}$$

We assume that N, ρ, u and p are smooth (at least C^2) and that N and ρ are both strictly positive. In our main result, Theorem 1.1, we show that equations (1.1)–(1.3) imply an elliptic equation on families of hypersurfaces, from which we expect infinite wave speeds. In light of such infinite speeds, we refer to (1.1)–(1.3) as the incompressible *covariant* Euler equations (instead of incompressible "relativistic" Euler equations).

THEOREM 1.1. *Assume* (N, ρ, u, p) *is a smooth solution of the incompressible covariant Euler equations,* (1.1)–(1.3)*, such that* $\rho \equiv \rho_0$ *is constant and* $p + \rho > 0$*. Then,* $\Phi \equiv \ln(\rho_0 + p)$ *satisfies*

$$\Delta_u\Phi + \dot{u}^\nu\partial_\nu\Phi + \nabla_\nu u^\mu\, \nabla_\mu u^\nu + R_{\mu\nu}u^\mu u^\nu = 0, \tag{1.6}$$

where $R_{\mu\nu}$ *is the Ricci tensor of the metric,* $\dot{u}^\nu \equiv u^\mu\nabla_\mu u^\nu$ *and*

$$\Delta_u\Phi \equiv \Pi^{\mu\nu}\nabla_\mu\partial_\nu\Phi$$

for $\Pi^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu$*. If the fluid flow is irrotational, then* (1.6) *is an elliptic PDE on each of the spacelike surfaces orthogonal to* u^μ*. Furthermore, if* $\rho > 0$ *is not constant,* p *satisfies Eq.* (2.10)*, which is again elliptic on each surface orthogonal to* u^μ *in the case of an irrotational fluid.*

Regarding the ellipticity stated in Theorem 1.1, the key observation is that $\Pi^{\mu\nu}$ is a Riemannian metric on each of the spacelike surfaces orthogonal to the fluid velocity u^μ. Thus Δ_u is the Laplace–Beltrami operator of the Riemannian metric $\Pi^{\mu\nu}$ on each of these surfaces. From the ellipticity of (1.6), we expect the incompressible covariant Euler equations (1.1)–(1.3) to exhibit infinite speed of propagation. In more detail, on any of the spacelike hypersurfaces orthogonal to u, boundary values determine solutions of (1.6) by standard elliptic theory [14]. So a signal, specifying boundary values, sent from one such hypersurface to a later such hypersurface, would determine Φ instantaneously throughout that surface. This