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Asymptotic stability and boundedness criteria for nonlinear retarded Volterra integro-differential equations

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ABSTRACT

In this article, we construct new specific conditions for the asymptotic stability (AS) and boundedness (B) of solutions to nonlinear Volterra integro-differential equations (VIDEs) of first order with a constant retardation. Our analysis is based on the successful construction of suitable Lyapunov–Krasovskii functionals (LKFs). The results of this paper are new, and they improve and complete that can be found in the literature.

1. Introduction

The Volterra integral equations (VIEs) and Volterra integro-differential equations (VIDEs) appeared after their establishment by Vito Volterra, in 1926. Thereafter they have wide applications in sciences and engineering. Namely, these equations appeared in many physical applications such as glass forming process, nanohydrodynamics, heat transfer, diffusion process in general, neutron diffusion and biological species coexisting together with increasing and decreasing rates of generating and wind ripple in the desert (see Wazwaz (2011)). More details about the sources where these equations arise can be found in physics, biology and engineering applications books. In addition, for more details of some of such applications, we referee the readers to the books of Burton (2005) and Wazwaz (2011). By this way we mean that it is worth and deserve to investigate properties of solutions of (VIDEs).

On the other hand, the important techniques used in the literature to search the qualitative behaviors (QBs) of paths of linear and non-linear (VIEs), (IDEs), (VIDEs), and etc., without finding the explicit solutions, are known as the second Lyapunov function(al) method, perturbation theory, fixed point method, the variation of constants formula and so on. In reality, we cannot find the analytical solutions of the equations mentioned, except very particular cases, and some time it become impossible to find the solutions, except numerically. Therefore it is an important need to use the former methods during the investigations.

Particularly, in the last four decades, researchers have produced a vast body of important results on the qualitative properties (QPs) of (VIDEs) by using the methods mentioned. In fact, several qualitative properties (QPs) of solutions; stability (S), boundedness (B), convergence (C), globally existence (GE) of solutions, etc., of different and the same models of linear and nonlinear (VIDEs) have been examined in the literature by many authors. For a comprehensive review and some recent results of (VIEs) and (VIDEs), we refer the reader to see (Atkinson, 1997; Becker, 2009; Brunner, 2004; Burton, 1979; Burton, 1982; Burton, 2005; Costarelli and Spigler, 2014; Furumochi and Matsuoka, 1999; Graef and Tunç, 2015; Graef et al., 2016; Hara et al.,1990; Hritonenko and Yatsenko, 2013; Maleknejad and Najafi, 2011; Miller,1971; Morchalo, 1991; Napoles Valdes, 2001; Raffoul, 2004; Raffoul, 2007; Raffoul, 2013; Staffans, 1988; Tunç, 2016a,b,c; Tunç, 2017a,b,c; Tunc and Ahyan, 2017; Tunç and Mohammed, 2017a,b; Tunç and Mohammed, 2017c; Vanualailai and Nakagiri, 2003; Zhang, 2005; Da Zhang, 1990 and the references therein). In that scientific sources, the authors obtained many interesting and valuable results on the (QPs) of specific (VIDEs). The mentioned authors benefited from the Lyapunov functions (LFs) or (LKFs) and obtained sufficient conditions which imply (S), (B), (C), etc. of solutions.

As renowned from this way, the following scientific works are notable.

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Morchalo (1991) considered the following scalar Volterra integro-differential equation

$$\frac{d}{dt}\left[x(t)-\int_0^t D(t,s)x(s)ds\right]=A(t)x(t)+\int_0^t C(t,s)x(s)ds \quad (1)$$

with $x(t_0)=x_0$, where $t_0 \geqslant 0$, $x \in \Re$, $A(t)$ is a continuous function for $t \in J$, $J=[0,\infty)$, and $C(t,s)$ and $D(t,s)$ are continuous functions for $0 \leqslant s \leqslant t < \infty$.

The author discussed the (B) of solutions of the (VIDE) (1) by means of a Lyapunov function. The assumptions are constructed in (Morchalo, 1991) are given below.

Assumptions A Morchalo, 1991. Let

$$Z(t,x(t))=x(t)-\int_0^t D(t,s)x(s)ds$$

and

$$a(t,k)=A(t)+k\int_t^\infty |C(u,t)|du+\frac{1}{2}\int_0^t |A(t)D(t,s)+C(t,s)|ds.$$

It is assumed that the following assumption are true.

$(M1)$ *There are positive constants* m and M *such that*

$$x^2 \leqslant mZ^2(t,x) \text{if} |x| \leqslant M, t \in J, J=[0,\infty).$$

$(M2)$ *There is a positive constant* m_1 *such that*

$$|A(t)D(t,s)| \leqslant m_1|C(t,s)| \text{for all} 0 \leqslant s \leqslant t < \infty.$$

$(M3)$ *There is a positive constant b such that the following integral is convergent*;

$$\int_0^t |C(t,s)|Z^2(s,x(s))ds \leqslant b < \infty \text{for all} t \in J, |x| \leqslant M.$$

$(M4)$ *There is a positive constant c such that the following integral is convergent*;

$$\int_0^\infty \left(\int_0^t |C(t,s)|Z^2(s,x(s))ds\right)dt \leqslant c < \infty.$$

$(M5)$ *There are positive constants* a and k *such that*

$$a(t,k) \leqslant -a < 0 \text{for all} t \in J,$$

and

$$\frac{1}{2}mm_1 - k \geqslant 0.$$

Theorem A Morchalo, 1991. *Let assumptions* (M1)–(M5) *be hold Then the solution* $x(t)=x(t,t_0,x_0)$ *of* (VIDE) (1) *is* f*-bounded.*

Besides, recently, Tunç (2017) considered the (VIDE) without delay of the form

$$\frac{d}{dt}\left[x(t)-\int_0^t D(t,s)x(s)ds\right]=-A(t)x(t)+\int_0^t C(t,s)x(s)ds + e(t,x) \quad (2)$$

with $x(t_0)=x_0$, where $t_0 \geqslant 0$, $x \in \Re$, $A(t)$ and $e(t,x)$ are continuous functions for $t \in J$, $J=[0,\infty)$, and $J \times \Re$, respectively, and $C(t,s)$ and $D(t,s)$ are continuous functions for $0 \leqslant s \leqslant t < \infty$. The author investigated the (AS) and (B) of solutions of (VIDE) (2) by defining new suitable Lyapunov functions.

In this paper, instead of (VIDEs) (1) and (2), we are concerned with the (QPs) of solutions of nonlinear first order retarded (VIDEs) equations of the form of

$$\frac{d}{dt}\left[x(t)-\int_{t-\sigma}^t b(t,s)g(x(s))ds\right]=-a(t)x(t)+\int_{t-\sigma}^t c(t,s)g(x(s))ds+p(t,x(t),x(t-\sigma)) \quad (3)$$

with $x(t_0)=x_0$, where $t-\sigma \geqslant 0$, σ is a positive constant, $x \in \Re$, $a(t)$, $g(x)$ and $p(t,x,x(t-\sigma))$ are continuous functions for $t \in \Re_+$, $\Re_+=[0,\infty)$, on $\Re$, and $\Re_+ \times \Re \times \Re$, respectively, and $b(t,s)$ and $c(t,s)$ are continuous functions for $0 \leqslant s \leqslant t < \infty$.

We assume throughout the paper that when we need x denotes $x(t)$, that is, $x=x(t)$.

For any $t_0 \geqslant 0$ and initial function $\phi \in C([t_0-\sigma,t_0])$, let $x(t)=x(t,t_0,\phi)$ denote the solution of (VIDE) (1) on $[t_0-\sigma,\infty)$ such that $x(t)=\phi(t)$ on $\phi \in C([t_0-\sigma,t_0])$.

Let $C[t_0,t_1]$ and $C[t_0,\infty)$ denote the set of all continuous real-valued functions on $[t_0,t_1]$ and $[t_0,\infty)$, respectively.

2. Stability and boundedness

Definition 2.1. Let $f:\Re_+ \to \Re$ be a continuous and non-negative function. The zero solution of

$$|y(t)-\int_{t-\sigma}^t b(t,s)y(s)ds| \leqslant f(t), y(t_0)=x_0 \quad (4)$$

for $t \in \Re_+$ is said to be

$(A1)$ $f-$ (S) if for given each $\varepsilon > 0$ and each $t_0 \geqslant 0$, there exists a $\delta=\delta(\varepsilon,t_0) > 0$ such that $\phi \in C[0,t_0]$, for all $t \in \Re_+$, $[|\phi| \leqslant \delta$ and $f(t) \leqslant \delta] \Rightarrow |y(t,t_0,\phi)| \leqslant \varepsilon$,

$(A2)$ asymptotic $f-$ (S) if it is $f-$ (S) and

$$\lim_{t\to\infty}|y(t,t_0,\phi)|=0$$

for every $|\phi| \leqslant \delta$ and every $f(t) \to 0$ as $t \to \infty$,

$(A3)$ $f-$ bounded if for every bounded function $f:\Re_+ \to \Re$, there exists a bounded solution $y(t,t_0,\phi)$ of (4).

Assumptions A.

Let

$$\omega_1(t,\mu_1)=a(t)-\frac{1}{2}\int_{t-\sigma}^t |a(t)b(t,s)+c(t,s)|ds - \mu_1\int_{t-\sigma}^\infty |c(u+\sigma,t)|du.$$

$(H1)$ There exist positive constants m_1 and m_2 such that

$$|a(t)b(t,s)| \leqslant m_1|c(t,s)| \text{for} 0 \leqslant s \leqslant t < \infty$$

and

$$|g(x)| \leqslant m_2|x| \text{for} x \in \Re.$$

$(H2)$ There exist positive constants μ_1 and k_1 such that

$$\omega_1(t,\mu_1) \geqslant k_1 > 0 \text{for all} t \in \Re_+.$$

Let $p(t,x,x(t-\sigma)) \equiv 0$.

Theorem 2.2. If assumptions (H1)–(H2) are true, *then all solutions of* (VIDE) (3) *are* f-bounded.