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Inclusion–exclusion by ordering-free cancellation

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ABSTRACT

Whitney's broken circuit theorem gives a graphical example of reducing the number of the terms in the sum of the inclusion–exclusion formula by a predicted cancellation. So far, the known cancellations for the formula strongly depend on the prescribed (linear or partial) ordering on the index set. We give a new cancellation method, which does not require any ordering on the index set. Our method extends all the 'ordering-based' methods known in the literature and in general reduces more terms. As examples, we use our method to improve some results on graph polynomials.

1. Introduction

Let $(\Omega, \mathscr{A}, \mu)$ be a measure space, P be a finite index set and $\{A_p\}_{p\in P} \subseteq \mathscr{A}$ a family of measurable sets. The formula

$$\mu\left(\bigcap_{p\in P} \overline{A}_p\right) = \sum_{I\subseteq P} (-1)^{|I|} \mu\left(\bigcap_{i\in I} A_i\right) \tag{1}$$

is known as the principle of inclusion–exclusion, where $\overline{A}_p$ is the complement of A_p.

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The principle of inclusion–exclusion is a classic counting technique in combinatorics and has been extensively studied [2,5,9–11,13]. Since the sum on the right side of Eq. (1) ranges over a large number of terms, it is natural to ask whether fewer terms would give the same result, that is, is it possible to reduce the number of terms by predicted cancellation? Lots of the answers to this question have been given by several authors. A well-known example is the one given by Whitney [13] in 1932 for chromatic polynomial of a graph, which states that the calculation of a chromatic polynomial can be restricted to the collection of those sets of edges which do not include any broken circuit as a subset.

Various cancellations for the inclusion–exclusion principle were given from the perspective of both combinatorics and graph theory in the literature. In [9], Narushima presented a cancellation for the inclusion–exclusion principle, depending on a prescribed ordering on the index set P. This result was later improved by Dohmen [2]. Using the same technique, Dohmen [5] also established an abstraction of Whitney's broken circuit theorem, which not only applies to the chromatic polynomial, but also to other graph polynomials, see [3–5,8,12] for details.

So far, the known cancellation methods for inclusion–exclusion principle strongly depend on the prescribed (linear or partial) ordering on the index set P. In this article we establish a new cancellation method, which does not require any ordering on P. Our method extends all the 'ordering-based' methods given in the previous literature and in general may reduce more terms. As examples, we use our 'ordering-free' method to improve results on the chromatic polynomial of hypergraphs, the independence polynomial and domination polynomial of graphs.

2. Inclusion–exclusion by predicted cancellations

For a subset B of a poset (partially ordered set) P, let B' denote the set of upper bounds of B which are not in B, that is,

$$B' = \{p \in P : p > b \text{ for any } b \in B\}.$$

In [9], Narushima presented a cancellation for the inclusion–exclusion principle on semilattices. This result was later extended to many forms. The following one was given by Dohmen [2]:

Theorem 2.1. *[2] Let $(\Omega, \mathscr{A}, \mu)$ be a measure space, P be a poset and $\{A_p\}_{p\in P} \subseteq \mathscr{A}$ a family of measurable sets. If $\mathfrak{X}$ is a class of subsets of P such that*

$$\bigcap_{p\in B} A_p \subseteq \bigcup_{p\in B'} A_p \tag{2}$$

for each $B \in \mathfrak{X}$. Then