Journal of Combinatorial Theory, Series A 162 (2019) 10–33

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Counting Steiner triple systems with classical parameters and prescribed rank

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ARTICLE INFO

Article history:
Received 19 September 2017
Available online xxxx

Keywords:
Steiner triple system
Linear code
Projective space
Affine space

ABSTRACT

By a famous result of Doyen, Hubaut and Vandensavel [6], the 2-rank of a Steiner triple system on $2^n - 1$ points is at least $2^n - 1 - n$, and equality holds only for the classical point-line design in the projective geometry $PG(n-1, 2)$. It follows from results of Assmus [1] that, given any integer t with $1 \leq t \leq n-1$, there is a code $C_{n,t}$ containing representatives of all isomorphism classes of STS$(2^n - 1)$ with 2-rank at most $2^n - 1 - n + t$. Using a mixture of coding theoretic, geometric, design theoretic and combinatorial arguments, we prove a general formula for the number of distinct STS$(2^n - 1)$ with 2-rank at most $2^n - 1 - n + t$ contained in this code. This generalizes the only previously known cases, $t = 1$, proved by Tonchev [13] in 2001, $t = 2$, proved by V. Zinoviev and D. Zinoviev [16] in 2012, and $t = 3$ (V. Zinoviev and D. Zinoviev [17], [18] (2013), D. Zinoviev [15] (2016)), while also unifying and simplifying the proofs.

This enumeration result allows us to prove lower and upper bounds for the number of isomorphism classes of STS$(2^n - 1)$ with 2-rank exactly (or at most) $2^n - 1 - n + t$. Finally, using our recent systematic study of the ternary block codes of Steiner triple systems [10], we obtain analogous results for the ternary case, that is, for STS(3^n) with 3-rank at most (or exactly) $3^n - 1 - n + t$.

We note that this work provides the first two infinite families of 2-designs for which one has non-trivial lower and upper

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https://doi.org/10.1016/j.jcta.2018.09.009

bounds for the number of non-isomorphic examples with a prescribed p-rank in almost the entire range of possible ranks.

1. Introduction

We assume familiarity with basic facts and notation concerning combinatorial designs [4] and codes [2], [7]. Throughout this paper, an incidence matrix of a design will have its rows indexed by the blocks, while the columns are indexed by the points of the corresponding design.

It was shown by Doyen, Hubaut and Vandensavel [6] that only the binary and ternary codes of Steiner triple systems can be interesting: for primes $p \neq 2, 3$, the GF(p)-code of any STS(v) has full rank v. The classical examples of STS are provided by the point-line designs in binary projective and ternary affine spaces. By a famous result of Doyen, Hubaut and Vandensavel, the 2-rank of a Steiner triple system on $2^n - 1$ points is at least $2^n - 1 - n$, and equality holds only for the classical point-line design in the projective geometry $PG(n-1, 2)$. An analogous result also holds for the ternary case, that is, for STS(3^n).

In [1], Assmus proved that the incidence matrices of all Steiner triple systems on v points which have the same 2-rank generate equivalent binary codes, and gave an explicit description of a generator matrix for such a code. In our recent systematic study of the binary and ternary block codes of Steiner triple systems [10], we also obtained a corresponding result for the ternary case. In all these cases, we give an explicit parity check matrix for the code in question.

Using these results, we will deal with the enumeration problem for STS on $2^n - 1$ or 3^n points with a prescribed 2-rank or 3-rank, respectively. In Section 2, we will use a mixture of coding theoretic, geometric, design theoretic and combinatorial arguments to prove a general formula for the number of *distinct* STS($2^n - 1$) with 2-rank at most $2^n - 1 - n + t$ contained in the relevant code. Our approach differs from the one used by the second author in [13] to find an explicit formula for the $STS(2^n - 1)$ of 2-rank $2^n - n$, and is somewhat reminiscent of the constructions of STS($2^n - 1$) with small 2-rank given by Zinoviev and Zinoviev [16,17], who also briefly mention a possible extension to higher ranks in [17]. However, our treatment will rely essentially on design theoretic and geometric methods, whereas [13], [16,17] use almost exclusively the language of coding theory. This allows us to give a unified, considerably shorter and, in our opinion, more transparent presentation.

The ternary case has not been studied before, except for our recent (mainly computational) work on STS(27) with 3-rank 24 [8]. In Section 3 – which is completely parallel to Section 2 – we provide general enumeration results also for the ternary case. Namely,